Introduction to Computational Geometry, Fall 2013

Please type your homework and submit it before the start of class (3:30pm) Tuesday, October 15, 2013. It's okay to hand draw the figures. It can also be submitted by email, but must be sent at least one hour before the start of class (2:30pm) October 15, 2013.

Homework 2

1. We saw in class that Thales theorem implies that given a line segment $\overline{ab}$, the set of all points x such that $\angle axb = 90^\circ$ is a circle. In this problem you will prove this algebraically. Recall that two vectors u and v are called *orthogonal* if they form a right angle. Equivalently, u and v are orthogonal if and only if $u \cdot v = 0$. I want you to prove that $\{x \in \mathbb{R}^2 \mid (x - a) \cdot (x - b) = 0\}$ is a circle. There are two things to prove. First, prove that if $(x - a) \cdot (x - b) = 0$ then x is on a circle. Be sure to tell me what the center and radius are. Second, prove that if x is on that circle, then $(x - a) \cdot (x - b) = 0$. It's possible to prove these at the same time, but make sure you prove both parts.
2. The *centroid* c of a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$ is defined as

   $$c = \frac{1}{n} \sum_{i=1}^{n} p_i.$$

   A. Prove that c is in the convex closure of P.

   B. Prove that if P is a set of points in the plane and c is in the convex hull (i.e. the boundary of the convex closure), then the points of P are collinear.

3. In class, we said that a set was convex if it was equal to its convex closure. An equivalent definition is that a set S is convex if and only if it contains the entire line segment $\overline{ab}$ for every pair of points $a, b \in S$.

   Let H be a halfspace in $\mathbb{R}^d$ defined as $H = \{x \in \mathbb{R}^d \mid x \cdot v \geq 0\}$ for some vector v. Prove that H is convex.

Prove that the cube $\{[x\ y\ z]^\top \in \mathbb{R}^3 \mid 0 \leq x, y, z \leq 1\}$ is convex.

4. Provide drawings of the following and answer the question at the end.
    - Draw the edges of a $3D$ cube as a planar straightline graph.
    - Draw its barycentric decomposition.
    - Draw the edges of an octahedron as a planar straightline graph.
    - Draw its barycentric decomposition.
    - Are the two barycentric decompositions isomorphic graphs? Why or why not?
5. **Extra Credit:** Let P be a set of n points in the plane. As in class, we can lift the points of P to the paraboloid in $\mathbb{R}^3$ by settting the z-coordinate of $p \in P$ to be the squared distance from p to the origin. That is,

$$p \in P \mapsto \begin{bmatrix} p_x \\ p_y \\ ||p||^2 \end{bmatrix}$$

We saw that the lower convex hull projects to the Delaunay triangulation. The upper convex hull also projects to a triangulation. What characterizes the triangles in the projection of the upper convex hull?