Planar Straight-line Graphs

Def A *graph* is a pair (V,E) where V is a finite set called vertices and E is a collection of pairs called edges in V (i.e. $E \subseteq \binom{V}{2}$).

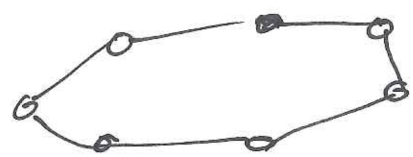
Examples of Graphs to Know

Don't say "click"

(1) Clique (Complete Graph) $E = \binom{V}{2}$

(2) Path

(3) Cycle



Def A *drawing* of $G=(V,E)$ *in the plane* is a representation of G s.t. vertices are distinct points in $\mathbb{R}^2$ and edges are simple arcs where

(1) arcs start and end at corresponding edge vertices.

(2) No arc touches a vertex other than its end points

(3) arcs only intersect at common end points.

Def A graph G is planar if there exists a drawing of G in the plane.

Def A planar straight-line graph (PSLG) is a drawing of a graph G in the plane where all edges are represented by straight line segments.

Thm [Fáry 1948] Every planar graph can be drawn as a PSLG.

This gives a geometric definition of planarity to complement our topological definition.

Examples
(non-crossing) Polygonal Chain - PSLG of a path
(simple) Polygon - PSLG of a cycle

PSLGs are easier to represent on a computer because we only need G and vertex positions.

Jordan Curve Thm for Polygons

Given a polygon P, $\mathbb{R}^2 \setminus P$ has 2 connected pieces (components) one is bounded, the other is unbounded, and P is the boundary of both.

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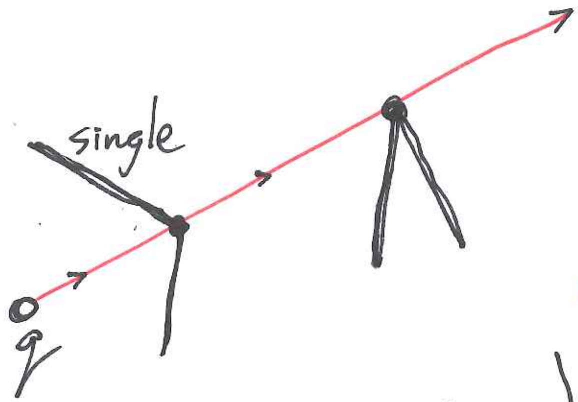
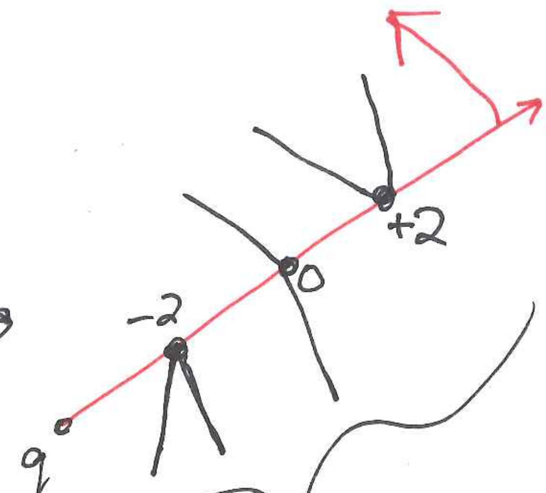
pf

(1) For any $q \in \mathbb{R}^2 \setminus P$, count the intersections between P and a ray emanating from q.



(2) Rotate the ray around q. The count only changes when we cross a vertex. Three cases



So count stays (3) odd or even. Cannot change parity.

$\Downarrow$

(4) define parity(q)

(5) Observe that if q, q' are connected by a straight line that doesn't touch P, then parity(q) = parity(q').

Jordan Curves give/define <u>faces</u> in a PSLG.

<u>Def</u> A <u>face</u> in PSLG $\mathcal{D}$ is a connected component of $\mathbb{R}^2 \setminus \mathcal{D}$.

The "infinite" or "unbounded" face counts as a face.

this is $\boxed{\text{not}}$ graph components

Euler's Formula

Let F be the set of faces of a PSLG.

$$|V| - |E| + |F| = 1 + \#\left\{\begin{matrix}\text{connected}\\ \text{components}\\ \text{of graph}\end{matrix}\right\}.$$