Last Time

Flips

Def Flippable

Def Locally Delaunay (LD)

Fact Not LD $\Rightarrow$ Flippable

Not LD LD

FlipToDel($\mathcal{T}$: triangulation of $P \subset \mathbb{R}^2$)

while $\exists e$ not LD

$\{$flip $e\}$

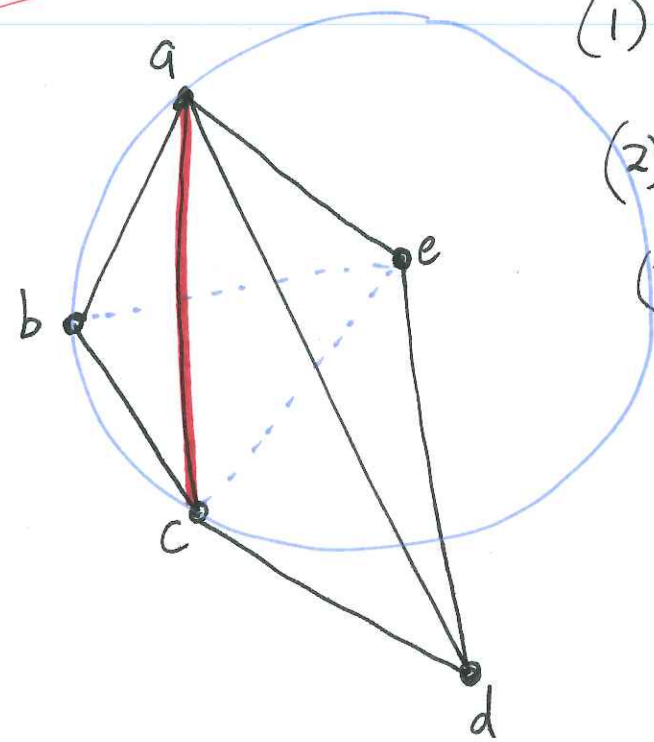
Questions: (1) Does it Terminate?

(2) Is the output Del_P?

the Delaunay Δn of P

A Closer Look at Locally Delaunay

LD $\not\Rightarrow$ Delaunay



(1) $\overline{ac}$ is LD but not Delaunay

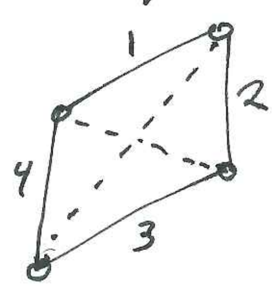
(2) $\overline{ad}$ is not LD

(3) $\overline{ac}$ is <u>not</u> LD after we flip $\overline{ad}$.

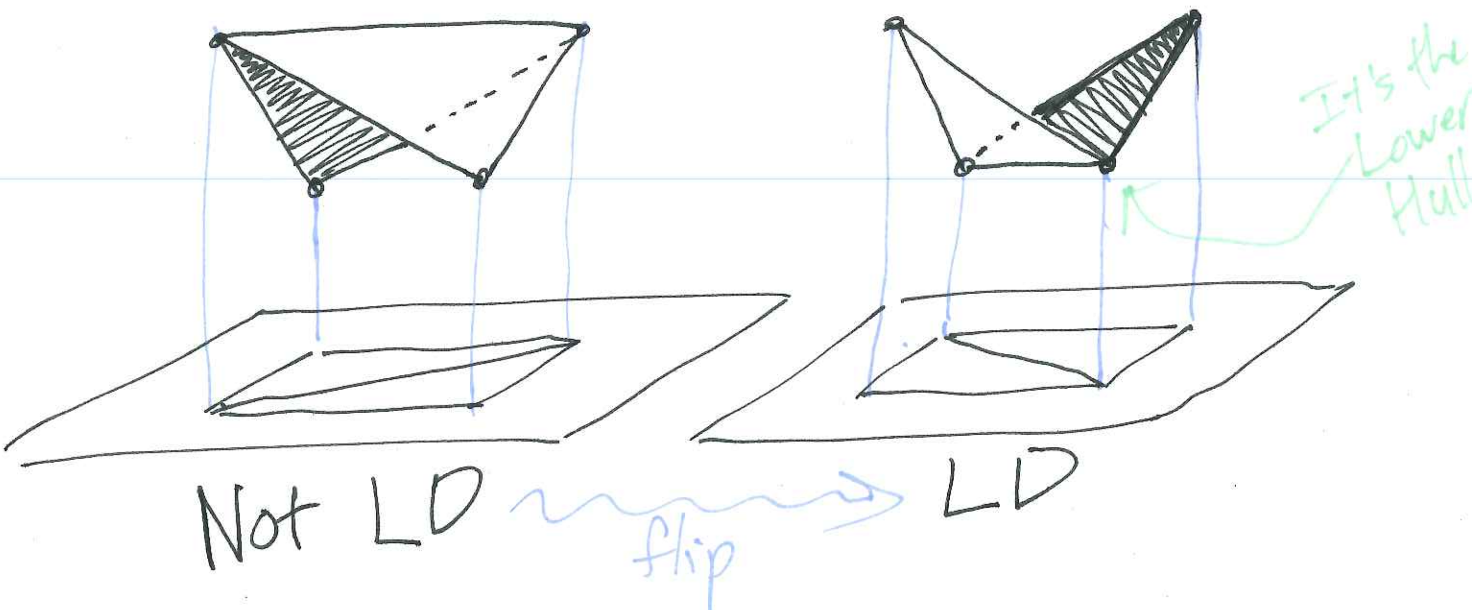
Local Conditions $\Rightarrow$ Constant time updates

Checking if an edge is LD takes one InCircle test.

At most 4 edges can change from LD to not LD.



Termination

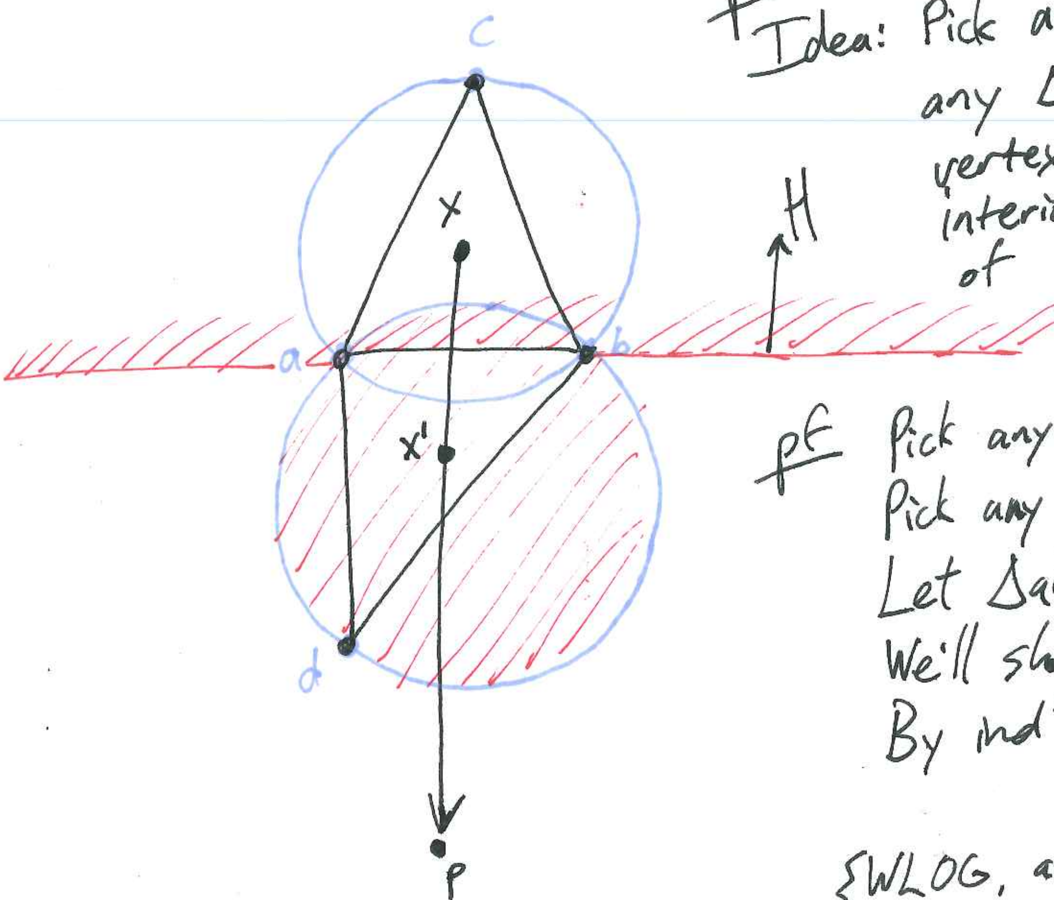


Each time we flip a non-LD edge, we decrease the volume below the "lifted triangulation."

Volume cannot go down forever. ⇒ Termination.

Count flips? Later.

Thm If T is a Δ^n of P s.t. all edges are LD, then $T = \mathrm{Del}_P$.



pf sketch

Idea: Pick any vertex and any Δ. Show the vertex is not in the interior of the circumcircle of the Δ.

pf Pick any $p \in P$.
Pick any $x \in CC(P)$
Let Δabc be Δ containing x.
We'll show $p \notin \mathrm{int}(\mathrm{circle}(abc))$.
By ind^n on $k = \#\{\text{edges crossed by } \overline{px}\}$

{WLOG, assume first edge is $\overline{ab}$.

i.e. $p \notin H$

Base $k=0$ is trivial.

Let d be vertex opposite c across $\overline{ab}$.

Pick $x' \in \overline{px} \cap \Delta abd$.

By ind^n, $p \notin \mathrm{circle}(abd)$.

$\overline{ab}$ is LD $\Longrightarrow$ $\boxed{\mathrm{circle}(abc) \subset H \cup \mathrm{circle}(abd).}$

$\Rightarrow p \notin \mathrm{circle}(abc)$.

Running Time

Counting Flips

Upper bound: $O(n^2)$

Why? Each edge is removed at most once.

For T a Δ^n, $h_T : CC(P) \to \mathbb{R}$ is the piecewise linear f^n we get by lifting vertices to the parabola and interpolating Δs.

If we flip $T \to T'$ then $h_{T'} \leq h_T$. $(*)$

If $x \in$ edge we flipped out, then

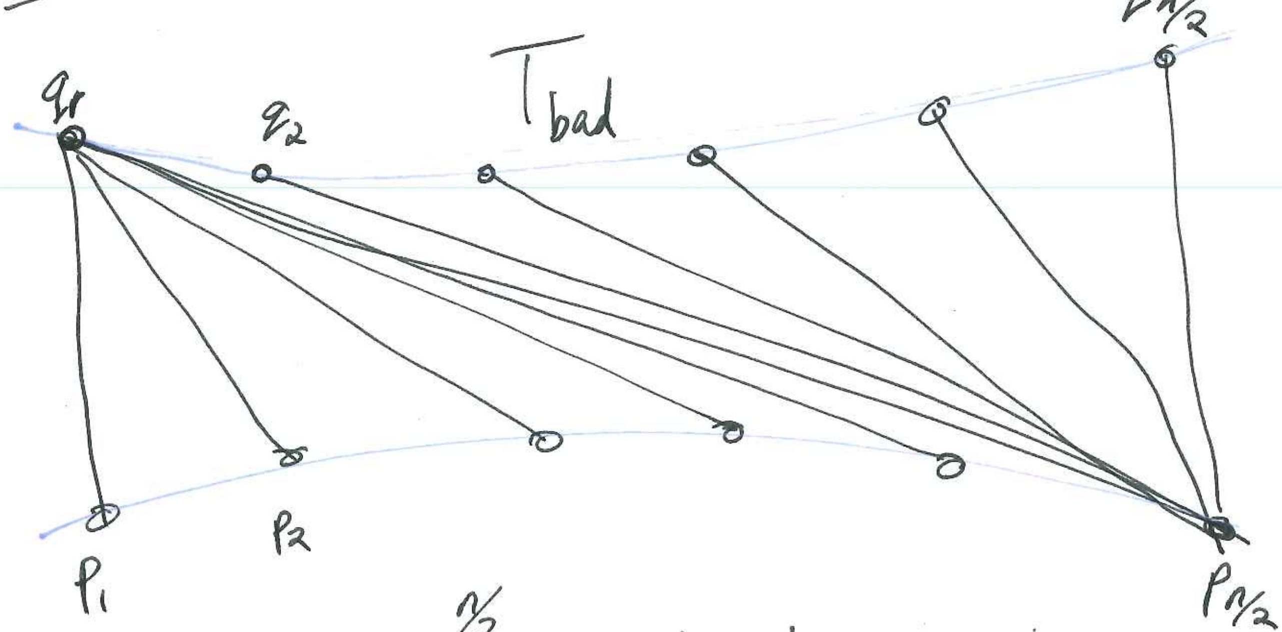
$$h_{T'}(x) < h_T(x).$$

If the edge appears later in T'' we have

$$h_{T''}(x) = h_T(x) > h_{T'}(x)$$

which contradicts $(*)$.

Lower Bound



$$\Phi(T) = \sum_{i=1}^{n/2} \min_{q_j \sim p_i} |i-j|$$

$$\Phi(Del(P)) = 0$$

$$\Phi(T_{bad}) = \sum_{i=1}^{n/2} (i-1) = \Theta(n^2)$$

$T \to T'$ by one flip

$$\Rightarrow |\Phi(T) - \Phi(T')| \leq 1$$

$\Rightarrow$ Flipping T_{bad} to $Del P$ requires $\Theta(n^2)$ flips.

Wrapup

Flips $\rightsquigarrow$ Greedy Flip Algorithm.

Edges that are not LD can be flipped.

If we only flip non-LD edge, we terminate.

If all edges are LD, we have ~~the~~ Del.

Running Time $O(n^2)$

Sometimes $O(n^2)$ flips are necessary.