Last Time

Greedy Flip Algorithm takes $O(n^2)$ time.

Any Δ^n can be flipped to Delaunay

Today Randomized Incremental Construction

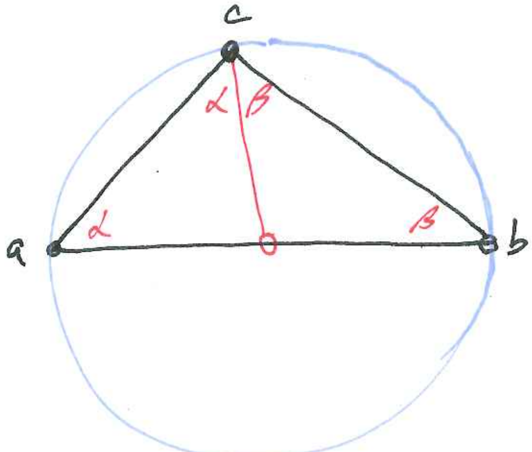
...but first, yet another definition of (characterization) the Delaunay Δ^n.

Claim: Among all Δ^ns of a point set P, the Delaunay Δ^n maximizes the minimum angle.

pf idea: Show that flipping an edge that is not LD can only increase the smallest angle.

So, our algorithmic result, gives a way to prove facts about DelP.

First, a refresher from high school/ancient Greece.

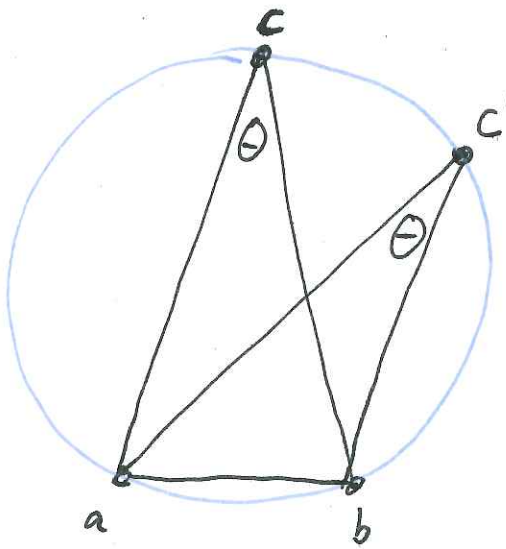


<u>Thales Thm</u> If $\overline{ab}$ is the diameter of circle(a,b,c), then $\angle acb$ is $90°$.

<u>pf</u> $2\alpha + 2\beta = 180°$

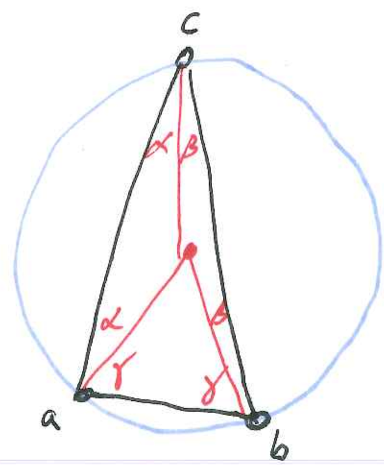
$\Rightarrow \angle acb = \alpha + \beta = 90°$.

This is true more generally.



If Δabc and $\Delta abc'$ have the same circumcircle, then $\angle acb = \angle ac'b$.

<u>pf</u>

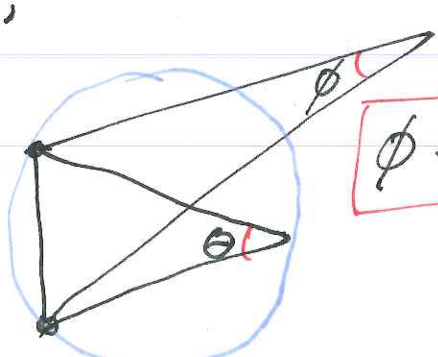


$2\alpha + 2\beta + 2\gamma = 180°$

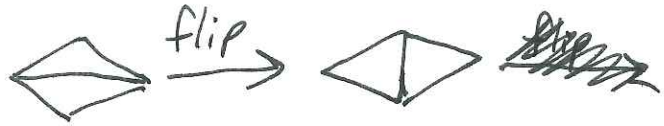
$\Rightarrow \angle acb = \alpha + \beta = 90 - \gamma$

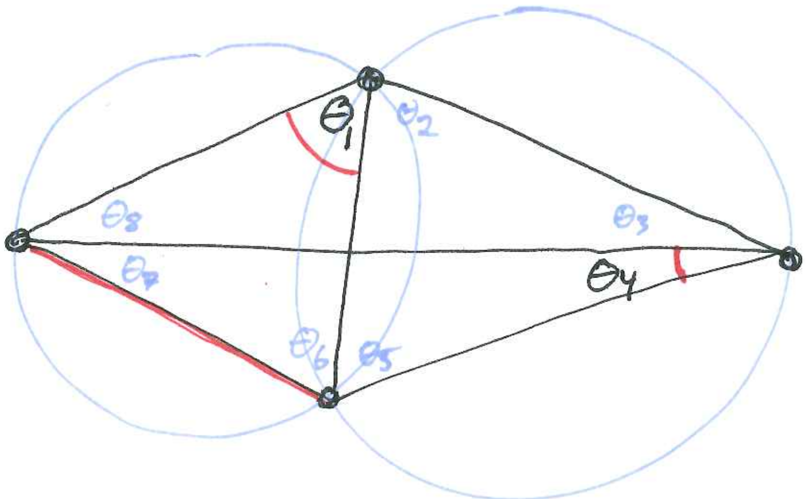
For c', α and β change but γ is the same so $\alpha + \beta$ does not change. Be Careful: Add <u>signed</u> angles.

So,



$\boxed{\phi < \theta}$





$\theta_1 > \theta_4$

$\theta_2 > \theta_7$

$\theta_6 > \theta_3$

$\theta_5 > \theta_8$

So, every angle after the flip is bigger than some angle that appeared before the flip.

Incremental Algorithms

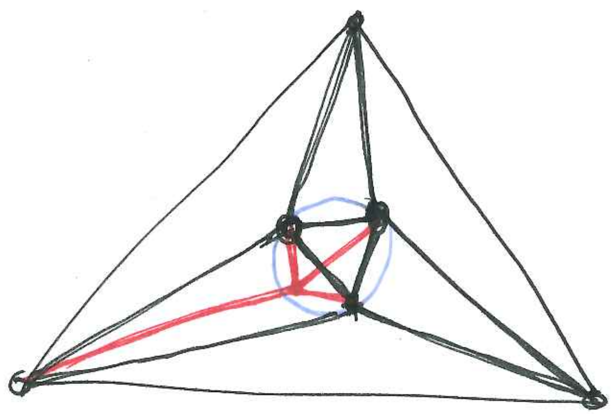
A staple of computational geometry

Idea: Add points one at a time.

Invariant: After i points added, we have computed the Del. Δ'n of those i pts.

1st Step Start with convex hull.

But for now assume its a triangle.



2nd Step Add one new point.

- Find Δ containing pt.

- Split it into 3
- Run the Greedy Flip Algorithm.

3rd Step Repeat.

Question: How many flips?

Some observations

1) All new triangles in step i have a vertex at p_i.

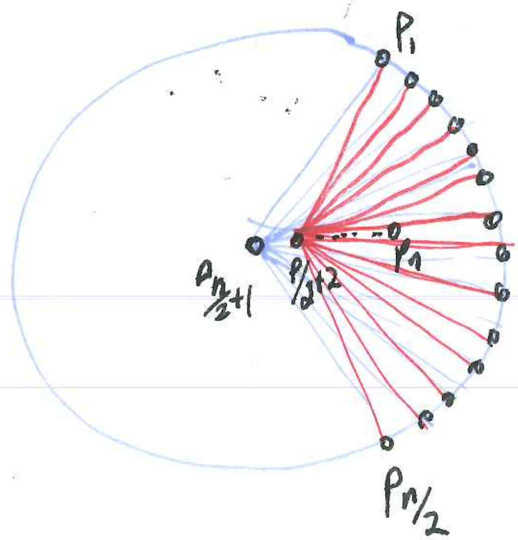
2) Every flip adds one edge incident to p_i

3) No flip removes an edge incident to p_i

$\Longrightarrow$ degree$(p_i) - 3$ flips in step i.

This is the degree in $Del_{\{p_1, \ldots, p_i\}}$.

This could still be quadratic.



First $n/2$ points "near" a circle.

Next $n/2$ points all require $n/2 - 2$ flips.

$\Rightarrow$ $\boxed{\Theta(n^2) \text{ total flips.}}$

Idea: Add the points in a random order

all permutations have equal prob.

Let $\langle p_1, \ldots, p_n \rangle$ be the random order.

Let $Q_i = \{p_1, \ldots, p_i\}$ (Q_i is the first i pts in the order)

We want to know the degree of p_i in Del_{Q_i}, call it δ_i.

$$\boxed{\text{We know } \#\text{flips} < \sum_{i=1}^{n} \delta_i}$$

What about $E[\#\text{flips}]$?

Expectation over all random orderings.

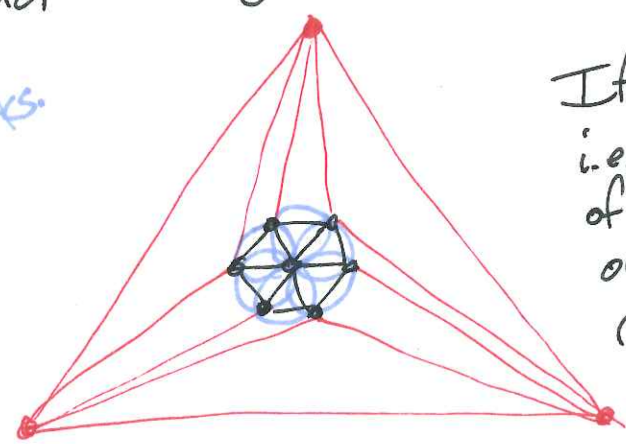
Use linearity of expectations:

$$E[\#\text{flips}] < E\left[\sum_{i=1}^{n} \delta_i\right] = \sum_{i=1}^{n} E[\delta_i]$$

Before we fix the algorithm, let's deal with the "big Δ" problem.

Idea 1: Add a big Δ of our own

This "often" works.



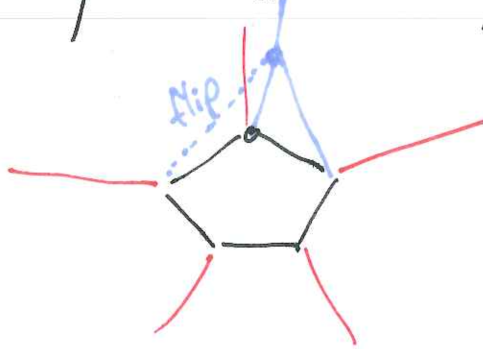
If it's "big enough" i.e. outside all circumcircles of input pts, then the output we want is contained in the output we get.

Question: What's big enough?

nearly collinear input pts

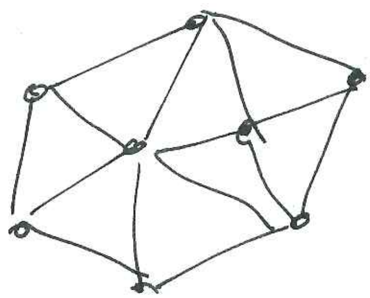
Circumcircle is huge.

Can we make the Δ infinitely large? add a point at infinity?



Need predicates to work with special ∞ points.

Observe: (1) Del_{Q_i} does not depend on the ordering of $p_1, \ldots, p_i$.



(2) Each point has equal probability of being p_i

(3) So, if we fix Q_i, $E[\delta_i]$ is the average degree of a vertex in Del_{Q_i}.

Recall Euler's Formula

$V - E + F = 2$

Since each face has at least 3 edges and each edge touches 2 faces

$3F \leq 2E$

So $V - E + \frac{2}{3}E \geq 2$

$\Rightarrow E \leq 3V$

(4) $\text{avg deg} = \frac{1}{i}\sum_{j=1}^{i} \deg_{Q_i}(p_j) = \frac{2}{i}E \leq \left(\frac{2}{i}\right)(3i) = 6$

(5) No matter what the first i points are, $E[\delta_i] \leq 6$

(6) $\sum_{i=1}^{n} E[\delta_i] \leq 6n$

(7) $O(n)$ flips.

I thought we expected $n \log n \ldots$