Voronoi Diagrams

Input: n points in $\mathbb{R}^2$, $P = \{p_1, \ldots, p_n\}$

Output: A polyhedral complex decomposing $\mathbb{R}^2$ with n cells, where cell V_i is

$$V_i = \{ x \in \mathbb{R}^2 : \|x - p_i\| \le \|x - p_j\| \ \forall j = 1 \ldots n \}$$

call this the Voronoi Cell of p_i

Voronoi Diagrams have a long history

Descartes 1644

Gauss + Dirichlet mid 1800s

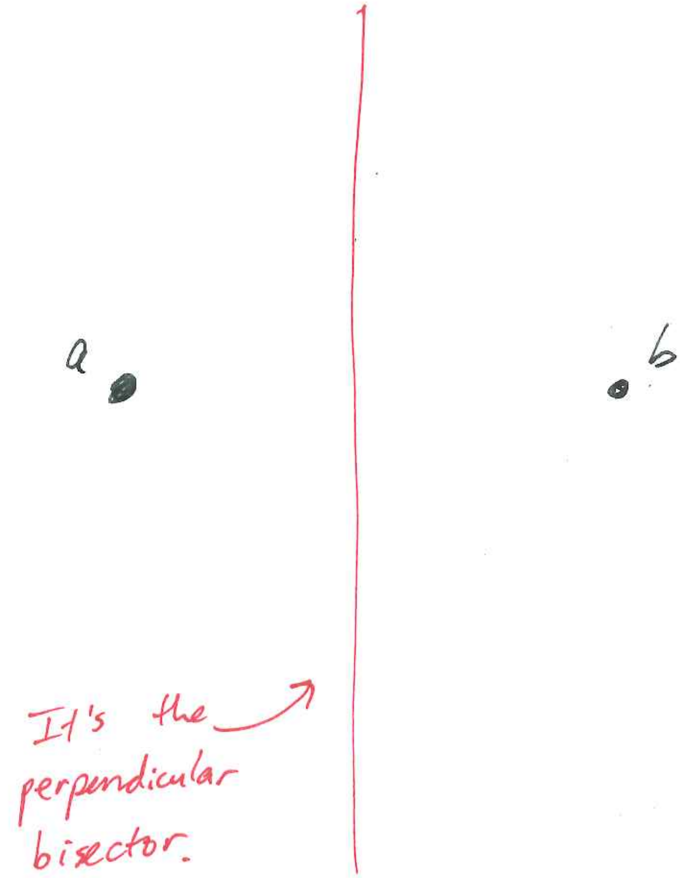
Voronoi 1908

Examples: Forest Fires
Fish Territories
Crystallography

Uses: Nearest Neighbor Classification (Post Office Problem)
Surface Reconstruction
Shape Analysis
VLSI Design

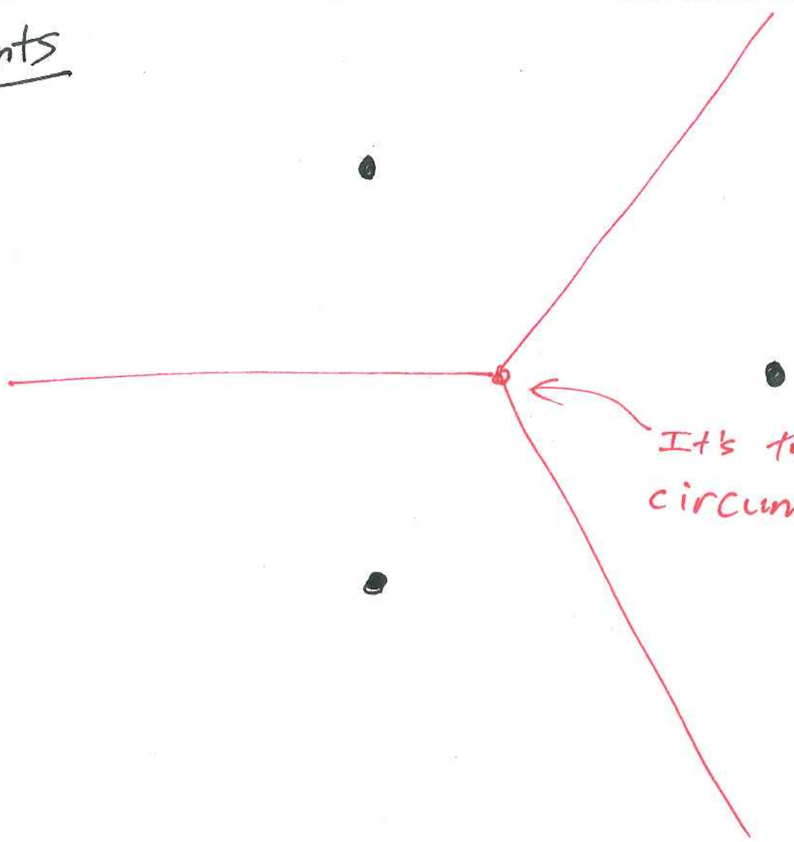
Warmup

2 points



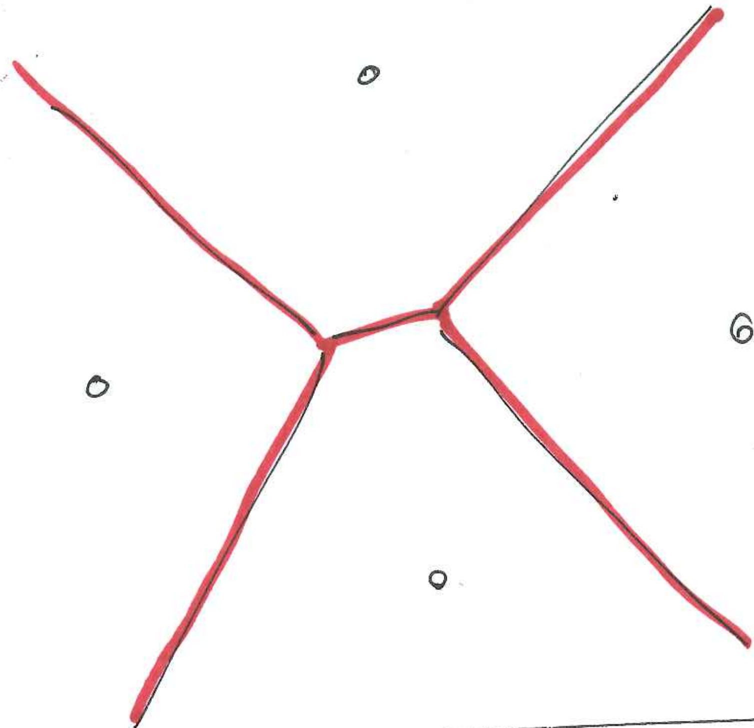
It's the perpendicular bisector.

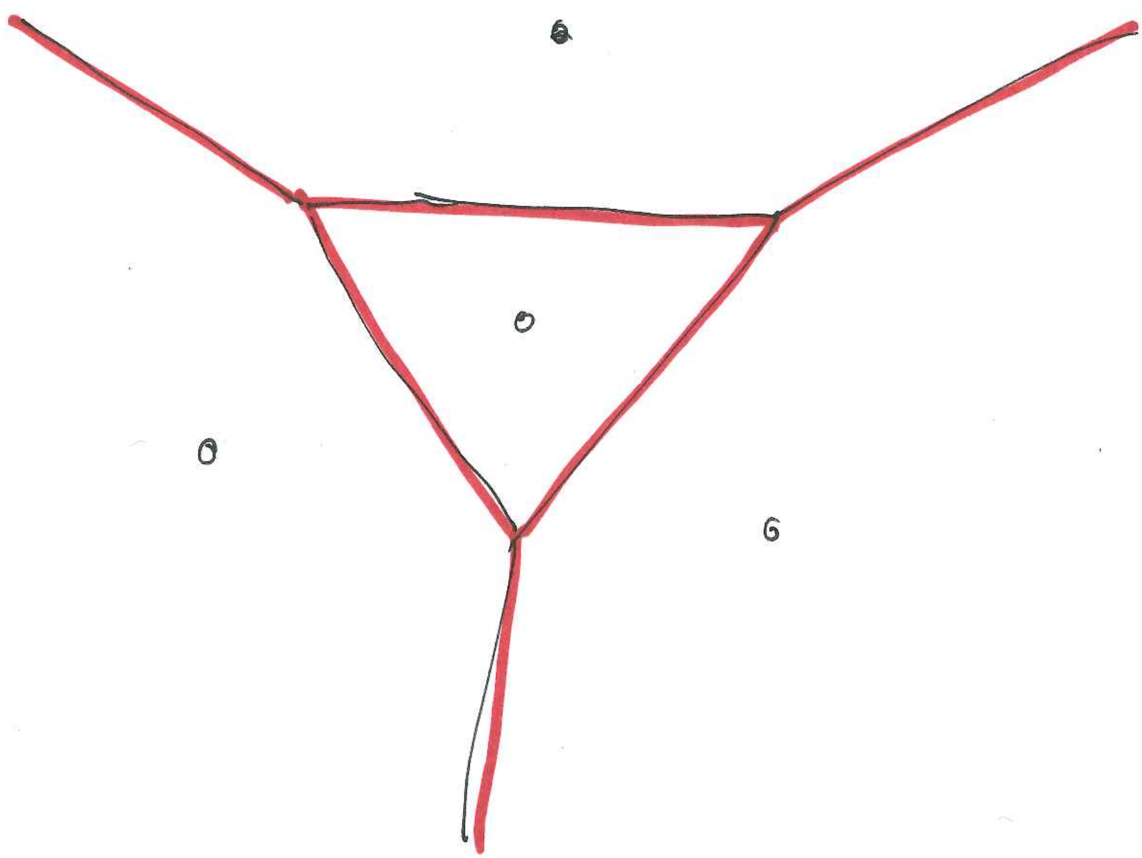
3 points



It's the circumcenter.

4 points





Claim: V_i is a polyhedron.

pf For a fixed p_j (s.t. $p_j \neq p_i$), the set $\{x \in \mathbb{R}^2 : \|x-p_i\| \leq \|x-p_j\|\}$ is a closed halfspace H_{ij} bounded by the perpendicular bisector between p_i and p_j.

Note that dimension is irrelevant here.

$$V_i = \{x \in \mathbb{R}^2 : \|x-p_i\| \leq \|x-p_j\| \ \forall j=1\ldots n\}$$

$$= \bigcap_{\substack{j=1 \\ (j \neq i)}}^{n} \{x \in \mathbb{R}^2 : \|x-p_i\| \leq \|x-p_j\|\}$$

$$= \bigcap_{j=1}^{n} H_{ij}$$

So, V_i is the intersection of a finite set of halfspaces. Therefore V_i is a polyhedron.

Cor: V_i is convex.

Claim: Vor_P is a polyhedral complex.

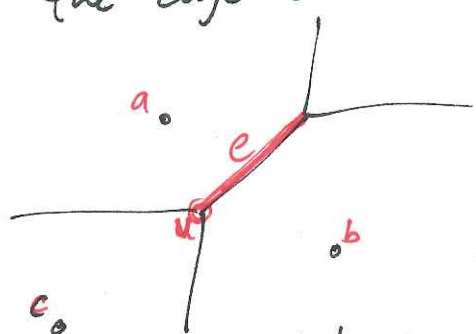
pf idea: We know each V_i is a polyhedron.

We need to show that any pair of faces intersect at a common subface.

But first, what are the faces? Polygons, Edges, Vertices

If a point is in more than one V_i then it is equidistant to more than one closest p_i.

In $\mathbb{R}^2$, the edge e is the set of points whose nearest point in P is either a or b, both are the same distance.



So, $e = V_a \cap V_b$

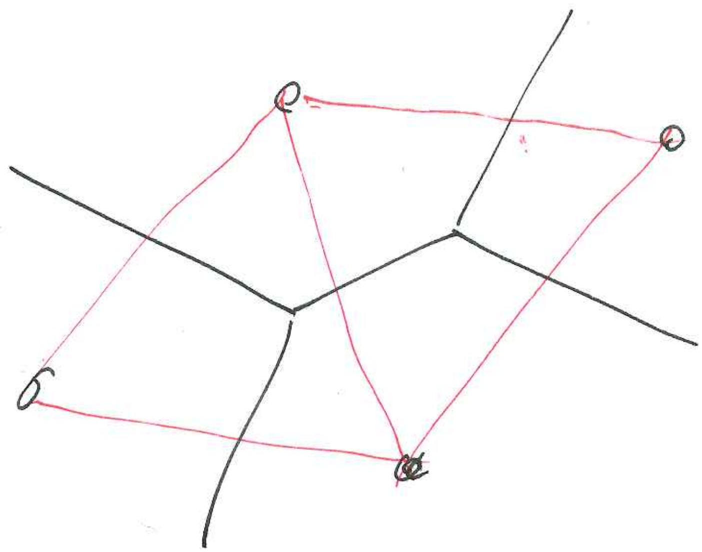
A vertex u has at least 3 nearest points in P.

$$u = V_a \cap V_b \cap V_c$$

If $x \in \bigcap_{i=1}^{k} V_i$ then $x \in \bigcap_{i \neq j = 1}^{k} \text{bisector}(p_i, p_j)$

This implies x is in a face of each V_i.

The Dual of the Voronoi Diagram is the Delaunay Triangulation.



Check, one vertex per face.
one triangle per Voronoi vertex
one edge per edge