Today's Lecture: More on projective duality.

Last Time:

In $\mathbb{R}^2$: point $\begin{bmatrix} p_x \\ p_y \end{bmatrix} \overset{\text{duality}}{\longleftrightarrow}$ line $y = 2p_x x - p_y$

slope $= 2p_x$ y-intercept $= -p_y$

In $\mathbb{R}^3$: point $\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \overset{\text{duality}}{\longleftrightarrow}$ plane $z = 2p_x x + 2p_y y - p_z$

To do higher dimensions, continue the pattern:

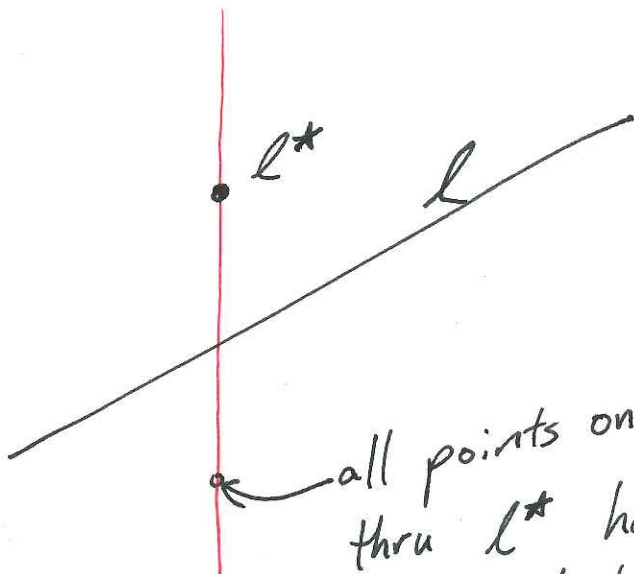
point $\begin{bmatrix} p \\ p_d \end{bmatrix} \longleftrightarrow$ hyperplane $x_d = 2p^T x - p_d$

$p \in \mathbb{R}^{d-1}$ $p_d \in \mathbb{R}$ $x_d \in \mathbb{R}$ $x \in \mathbb{R}^{d-1}$

Exercise 1

Given a (nonvertical) line ℓ in the plane.

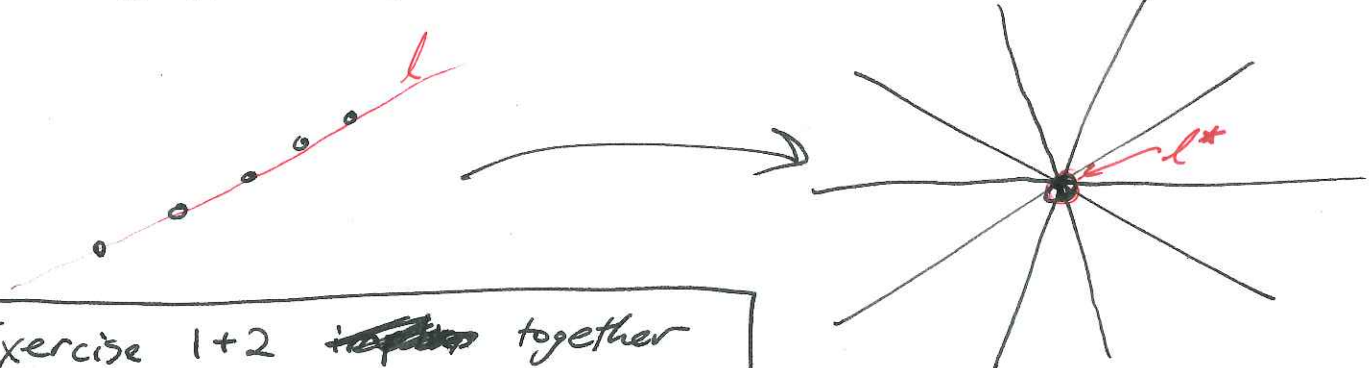
Describe the dual of the set of all lines parallel to ℓ



all points on the vertical line thru ℓ^* have the same slope in the dual as ℓ and thus are parallel to ℓ.

Exercise 2

Given a collection of collinear points, what can be said about the dual collection of lines.

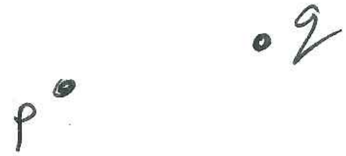


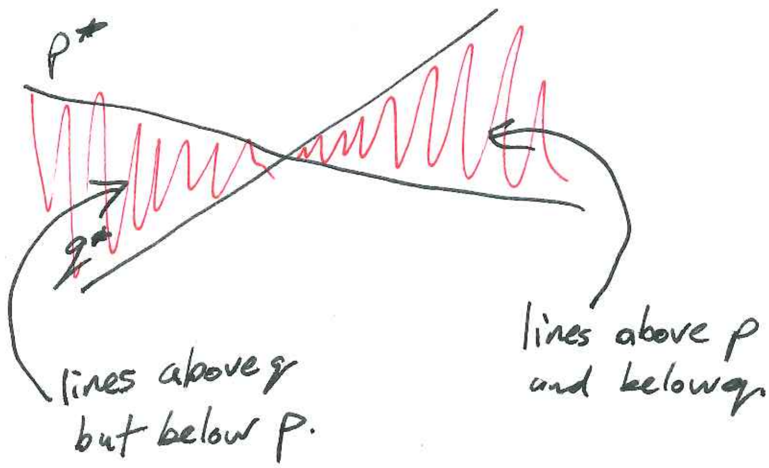
Exercise 1+2 together imply that the dual to a vertical line should be a point at infinity.

The dual lines all pass thru ℓ^*.

Exercise 3

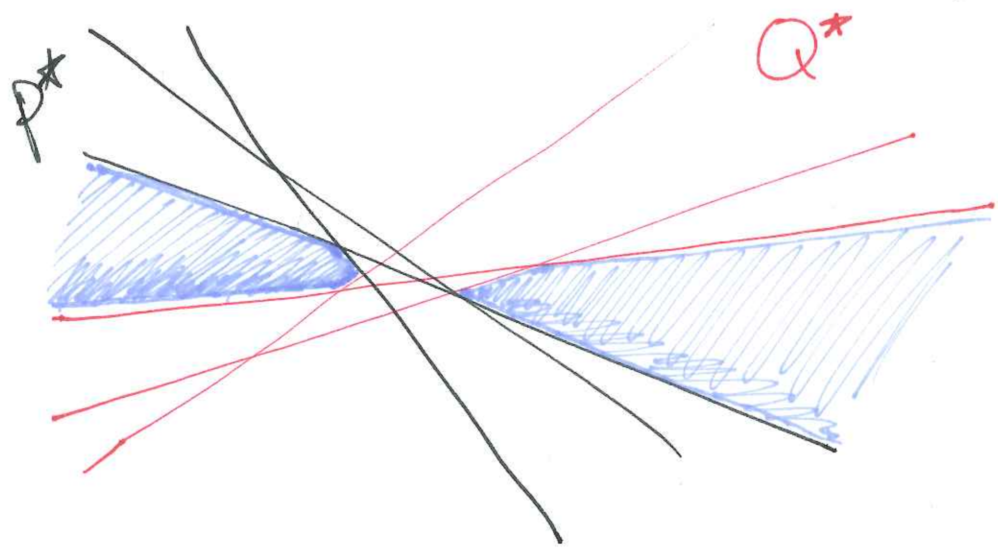
Given two points p, q, use duality to describe the set of lines passing between them.





Exercise 4

Given two sets of points P and Q, use duality to describe the set of lines separating P and Q.



Note: 2 convex pieces.

Can it be only 1?

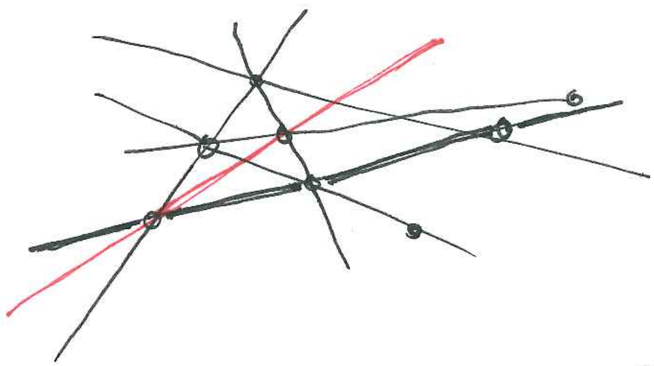
Can it be bounded?

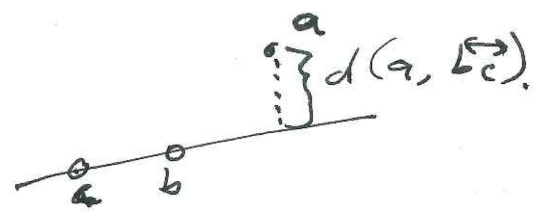
- What if no vertical line ~~passes~~ separates P and Q? Is it convex?

Dualizing Theorems:

Example (The Sylvester-Gallai Theorem)

Given n points in the plane that are not all collinear, there exists a line that passes through exactly 2 of the points.



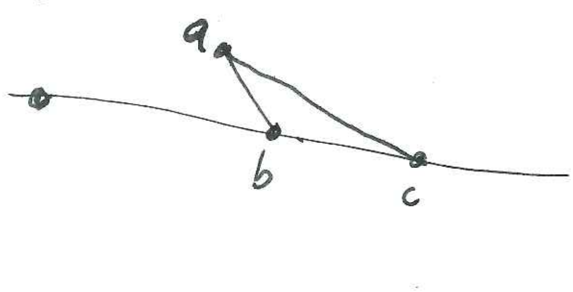


First, a proof.

Let $d(a, \overleftrightarrow{bc})$ denote the distance from a point a to the line $\overleftrightarrow{bc}$.

Pick a, b, c to be the triple of points not all on a line that minimizes $d(a, \overleftrightarrow{bc})$.

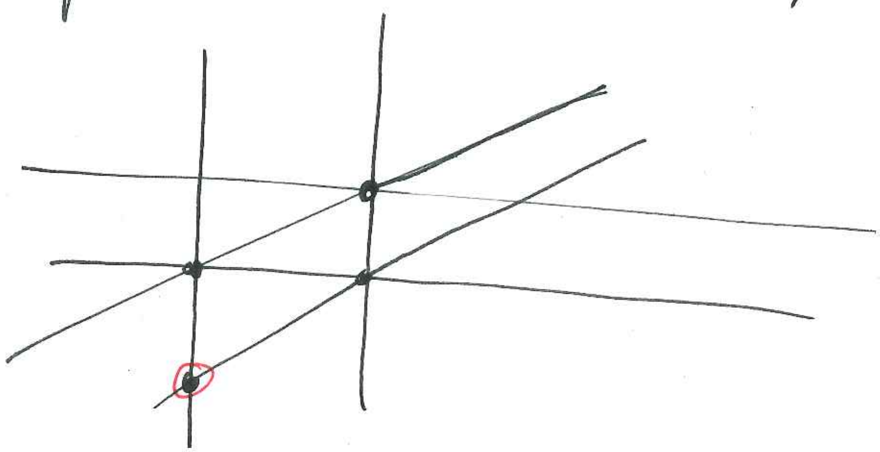
Suppose for contradiction that line $\overleftrightarrow{bc}$ contains at least 3 points. Then we can choose b and c so that $\angle abc \geq 90°$.



This implies $d(b, \overleftrightarrow{ac}) < d(a, \overleftrightarrow{bc})$, a contradiction.

Dual Version of Sylvester-Gallai

Given n lines, not all passing thru a common point, then there exists a point contained in exactly 2 lines.



Can you dualize the proof?