# Minimum Enclosing Ball (MEB)

Input: $n$ points $P \subset \mathbb{R}^d$

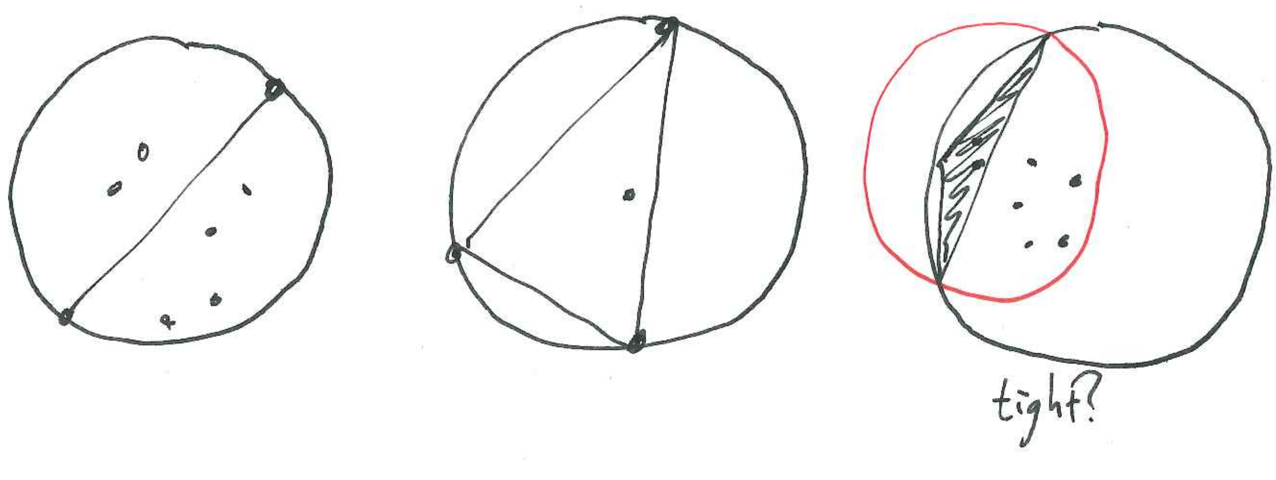
Output: center and radius of the smallest closed ball $B$ s.t. $P \subset B$.

$$\text{center}(P) = \underset{x \in \mathbb{R}^d}{\text{argmin}} \max_{p \in P} \|x - p\|$$

$$\text{radius}(P) = \max_{p \in P} \|p - \text{center}(P)\|$$

$$\text{ball}(P) = \text{ball}(\text{center}(P), \text{radius}(P))$$

Examples



Note that MEB is not a <u>linear program</u>.

It's a <u>quadratic program</u>.

Find $x, r$ to minimize $\cancel{\max_{p \in P} \|x-p\|}$ $r^2$

subject to

$$\|x-p_1\|^2 \le r^2$$

$$\vdots$$

$$\|x-p_n\|^2 \le r^2$$

Note: We square everything to avoid square roots (like Pythagoras).

<u>Fact</u> Center$(P)$ is the circumcenter of some $S \subseteq P$ and center$(P) \in \text{conv}(P)$.

# Carathéodory's Thm

If $x \in \text{conv}(P)$ then there is a subset $S \subseteq P$ with $|S| \leq d+1$ such that $x \in \text{conv}(S)$.

pf

Note: The usual proof is algebraic. One writes down the convex combination for $x$. Then one uses the ~~linear~~ affine dependence between the points to reduce the number of nonzero coefficients by one until at most $d+1$ remain.

A short proof. Delp decomposes conv(P) into simplices. Let $S$ be the vertices of ~~the~~ a simplex in Delp containing $x$. If there is degeneracy, perturb the points by moving them away from $x$.

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Why does this matter for MEB?

It means that $d+1$ points is always enough to describe the MEB.

# An algorithm!

## Randomized incremental construction!

A little twist: The input is two sets $P$ and $Q$ where the points of $Q$ are guaranteed to be on the boundary of ball $(P \cup Q)$.

$MEB(P, Q)$

If $P = \emptyset$ return circumcenter$(Q)$

$B_0 = MEB(\emptyset, Q)$

Let $n = |P|$

Randomly order $P$ as $p_1, \dots, p_n$.

For $i = 1$ to $n$

- If $p_i \in B_{i-1}$ set $B_i := B_{i-1}$
- If $p_i \notin B_{i-1}$ set $B_i := MEB(\{p_1, \dots, p_{i-1}\}, Q \cup \{p_i\})$

# Analysis of the Randomized Incremental MEB algorithm.

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As always, we want to analyze the <u>expected</u> running time of randomized incremental algorithms.

Let $T_{n,j}$ be the expected running time for $|P|=n$ and $j = d+1-|Q|$

$j$ is the number of bounding points we still need to find.

The main loop has 2 cases: $p_i \in B_{i-1}$ and $p_i \notin B_{i-1}$. The first case takes constant time. The second requires a recursive call.

By backwards analysis, we make the recursive call with probability $j/i$. So,

$$T_{n,j} = cn + \sum_{i=1}^{n} j/i \, T_{i-1,j-1}$$

Assume by induction that $T_{k,\ell} \leq f_\ell k$ for all $k \leq n$, where $f_\ell$ is a function of $\ell$ and $c$. Choose $f_\ell = c + j f_{j-1}$.

So, $T_{n,j} \leq cn + j f_{j-1} \sum_{i=1}^{n} \frac{i-1}{i} \leq (c + j f_{j-1}) n = f_j n.$

So, exp. running time is $O(n)$. Constant $f_j$ is superexponential in $d$.