Last Time

Real RAM — assume this
- arithmetic on coords

Points vs. Vectors

Linear Predicates
- comparison model.

use this

$$\underbrace{\operatorname{sign}\left(\det\begin{bmatrix} a & b & c \\ 1 & 1 & 1 \end{bmatrix}\right)}_{\text{orientation of } \Delta abc}$$

In General:

$$\operatorname{sign}\left(\det\begin{bmatrix} p_0 & \cdots & p_d \\ 1 & \cdots & 1 \end{bmatrix}\right)$$

Why?

- Avoids Division
- Only need sign (tricks available)
- Filtering Predicates
  - Use exact arithmetic packages only in "tricky" cases, i.e, det is very close to zero.

Computing Determinants

$$\mathrm{Det}(M) = \sum_{\substack{\sigma : \text{perm} \\ \text{of } [n]}} (-1)^{\operatorname{sgn} \sigma} \prod_{i=1}^{n} M_{i\sigma(i)}$$

$n \times n$

In 3D, you may have seen

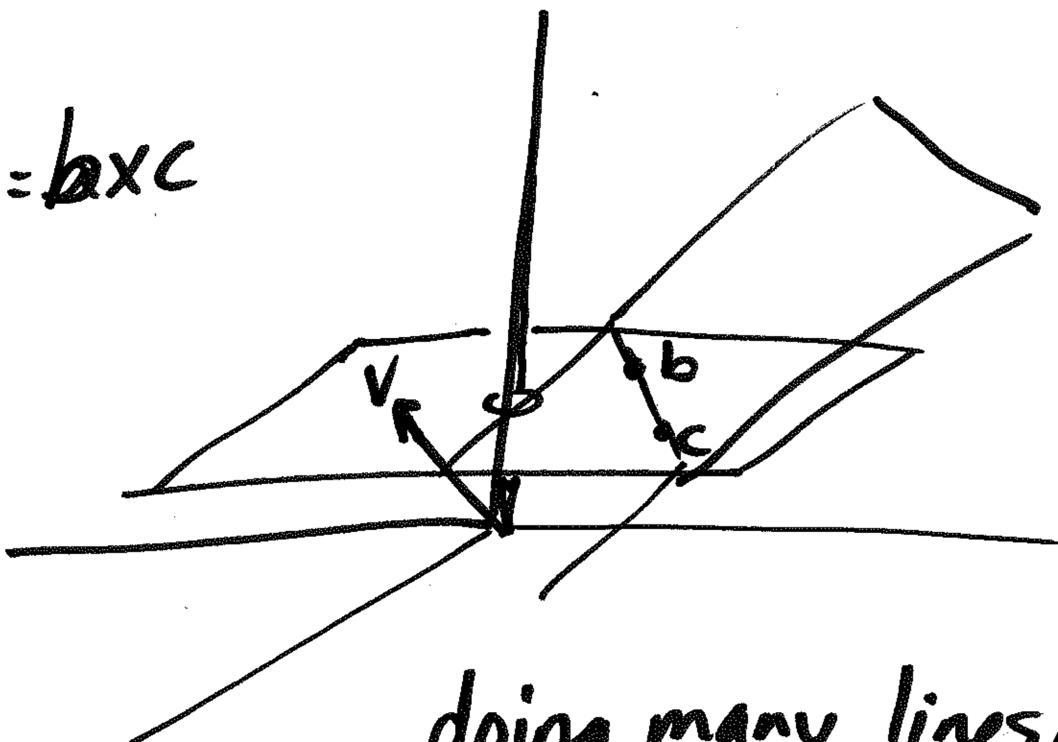
$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \cdot \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

Relation to Cross Product

$b, c \in \mathbb{R}^3$ $\qquad b \times c = \det \begin{bmatrix} \hat{i} & \uparrow & \uparrow \\ \hat{j} & b & c \\ \hat{k} & \downarrow & \downarrow \end{bmatrix}$

So, $a^T (b \times c) = \det \begin{bmatrix} a & b & c \end{bmatrix}$

$v = b \times c$



doing many lineside tests with $\overline{bc}$ are easier using dot prod. w/ v.

Properties

$$\det(AB) = \det(A)\det(B)$$

$$\det\begin{bmatrix} a & b & c \end{bmatrix} = -\det\begin{bmatrix} a & c & b \end{bmatrix}$$

Claim: $\det\begin{bmatrix} b-a & c-a \end{bmatrix} = \det\begin{bmatrix} a & b & c \\ 1 & 1 & 1 \end{bmatrix}$

<u>pf</u>

$$\begin{bmatrix} a & b & c \\ 1 & 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & a_x \\ 0 & 1 & a_y \\ 0 & 0 & 1 \end{bmatrix}}_{\det = 1} \begin{bmatrix} 0 & b_x-a_x & c_x-a_x \\ 0 & b_y-a_y & c_y-a_y \\ 1 & 1 & 1 \end{bmatrix}$$

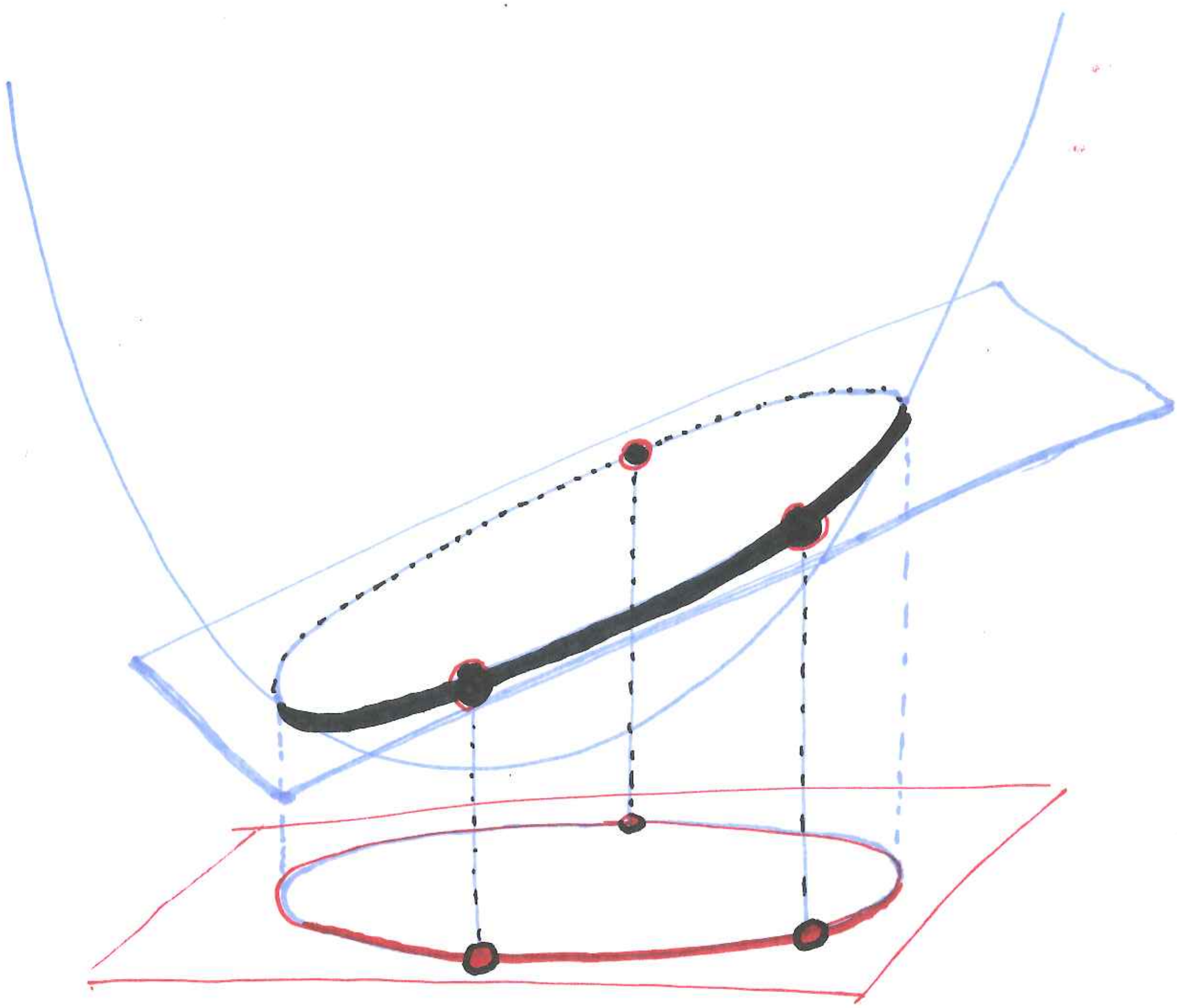
$$\Rightarrow \det\begin{bmatrix} a & b & c \\ 1 & 1 & 1 \end{bmatrix} = \det\begin{bmatrix} 0 & b_x-a_x & c_x-a_x \\ 0 & b_y-a_y & c_y-a_y \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \det\begin{bmatrix} b_x-a_x & c_x-a_x \\ b_y-a_y & c_y-a_y \end{bmatrix}$$

$$= \det\begin{bmatrix} b-a & c-a \end{bmatrix}$$

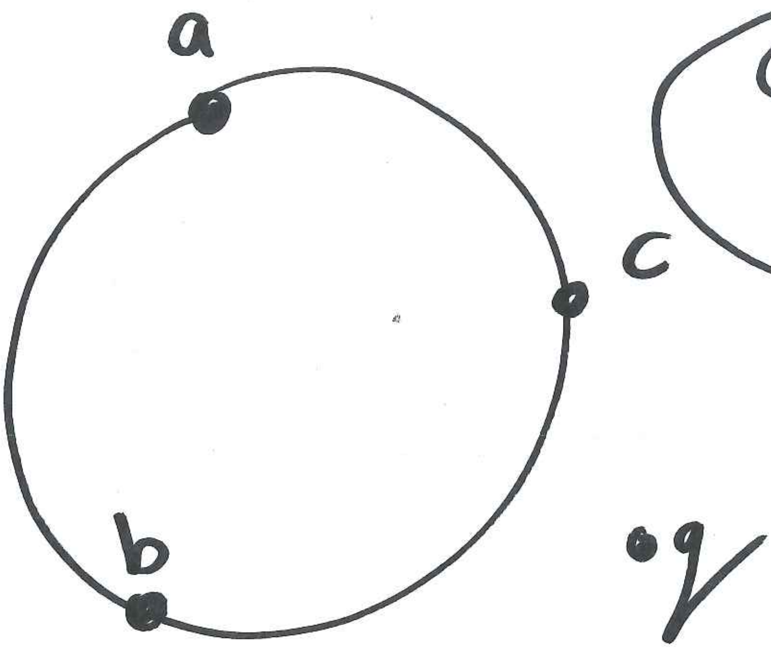
$a \mapsto \begin{bmatrix} a \\ 1 \end{bmatrix}$ This is a in "homogeneous coordinates"

InCircle Test



Lift the points to a paraboloid $\Pi = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : z = x^2 + y^2 \right\}$

$$\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \longmapsto \begin{bmatrix} x \\ y \\ x^2+y^2 \end{bmatrix} \in \mathbb{R}^3$$



$\bigcirc abc :=$ circumcircle of Δabc.

$$\mathrm{InCircle}(a,b,c,q) = \begin{cases} 1 & \text{if } q \text{ inside } \bigcirc abc \\ 0 & \text{if } q \text{ on } \bigcirc abc \\ -1 & \text{if } q \text{ outsid } \bigcirc abc \end{cases}$$

$$= \left(\frac{\operatorname{sign} \det \begin{bmatrix} a & b & c & q \\ \|a\|^2 & \|b\|^2 & \|c\|^2 & \|q\|^2 \\ 1 & 1 & 1 & 1 \end{bmatrix}}{\operatorname{sign} \det \begin{bmatrix} a & b & c \\ 1 & 1 & 1 \end{bmatrix}} \right)$$

Normalize by orientation of Δabc

Let H be the plane thru $\begin{bmatrix} a \\ \|a\|^2 \end{bmatrix}, \begin{bmatrix} b \\ \|b\|^2 \end{bmatrix}, \begin{bmatrix} c \\ \|c\|^2 \end{bmatrix}$

Let $v' = \begin{bmatrix} v \\ -\frac{1}{2} \end{bmatrix}$ be normal to H

So $H = \{ q' : q' \cdot v' = s \}$ for some s.

$s = v' \cdot \begin{bmatrix} a \\ \|a\|^2 \end{bmatrix} = v' \cdot \begin{bmatrix} b \\ \|b\|^2 \end{bmatrix} = v' \cdot \begin{bmatrix} c \\ \|c\|^2 \end{bmatrix}$

Suppose $p' = \begin{bmatrix} p \\ \|p\|^2 \end{bmatrix} \in H \cap \Pi$

then $\begin{bmatrix} p \\ \|p\|^2 \end{bmatrix} \cdot \begin{bmatrix} v \\ -\frac{1}{2} \end{bmatrix} = s$

$$\begin{aligned} & \| \\ & p \cdot v - \tfrac{1}{2}\|p\|^2 = s \\ & 2s = 2(p \cdot v) - \|p\|^2 \end{aligned}$$

$$\|p - v\| = \sqrt{\|p\|^2 - 2(p \cdot v) + \|v\|^2} = \underbrace{\sqrt{\|v\|^2 - 2s}}_{\text{indep. of } p}$$

Thus v is the circumcenter

and $\sqrt{\|v\|^2 - 2s}$ is the circumradius.