

Voronoi Diagrams

Input: n points in \mathbb{R}^2 , $P = \{p_1, \dots, p_n\}$

Output: A polyhedral complex decomposing \mathbb{R}^2 with n cells, where cell V_i is

$$V_i = \{x \in \mathbb{R}^2 : \|x - p_i\| \leq \|x - p_j\| \forall j = 1, \dots, n\}$$

↖ call this the Voronoi Cell of p_i

Voronoi Diagrams have a long history

Descartes 1644

Gauss + Dirichlet mid 1800s

Voronoi 1908



Examples: Forest Fires
Fish Territories
Crystallography

Uses: Nearest Neighbor Classification (Post Office Problem)
Surface Reconstruction
Shape Analysis
VLSI Design

Warmup

2 points

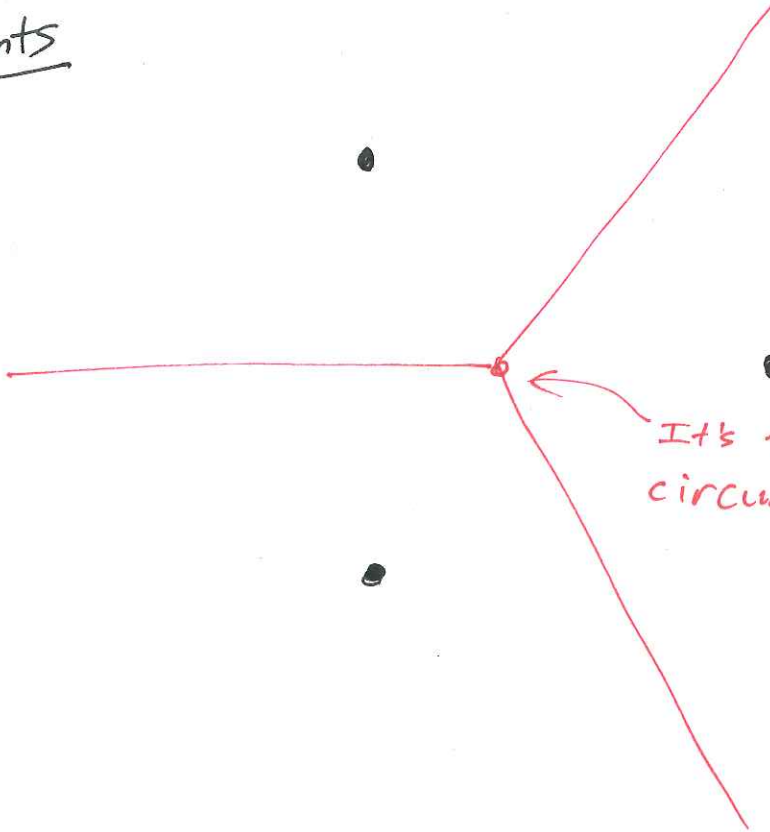
a

b

It's the
perpendicular
bisector.

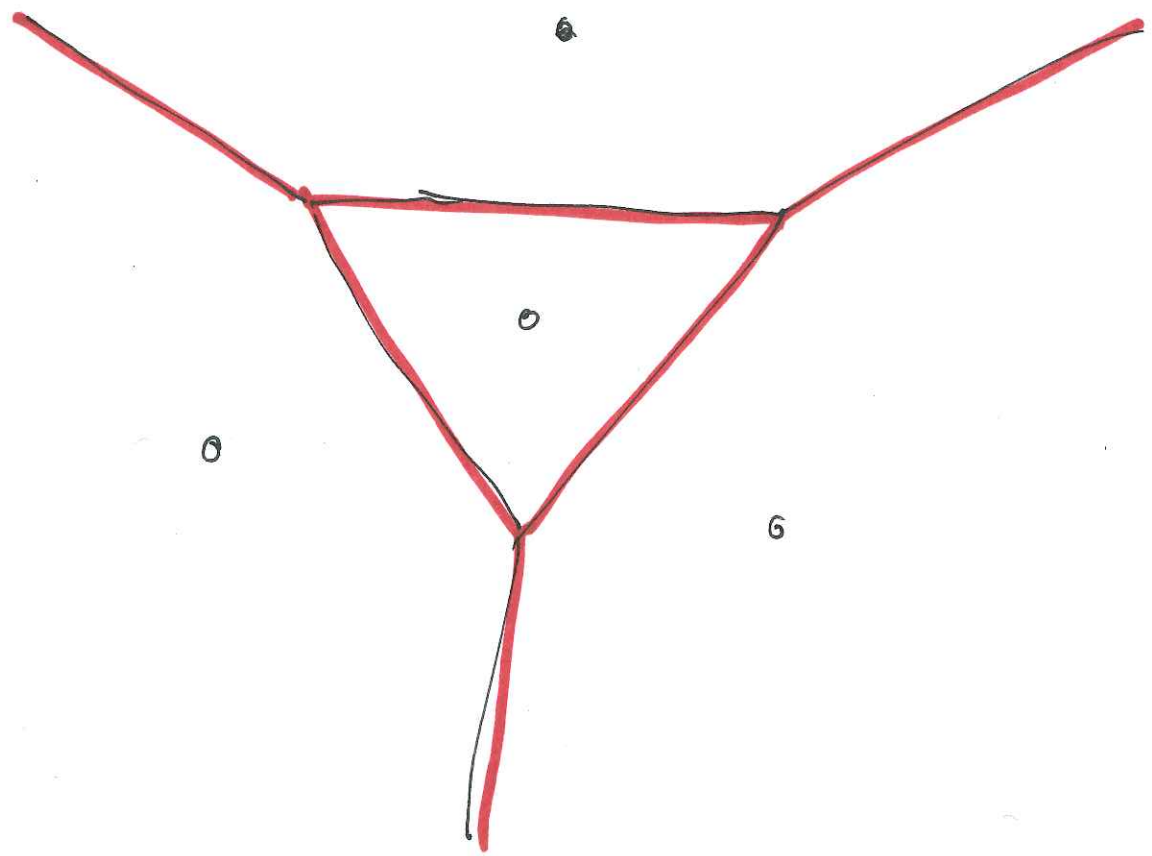
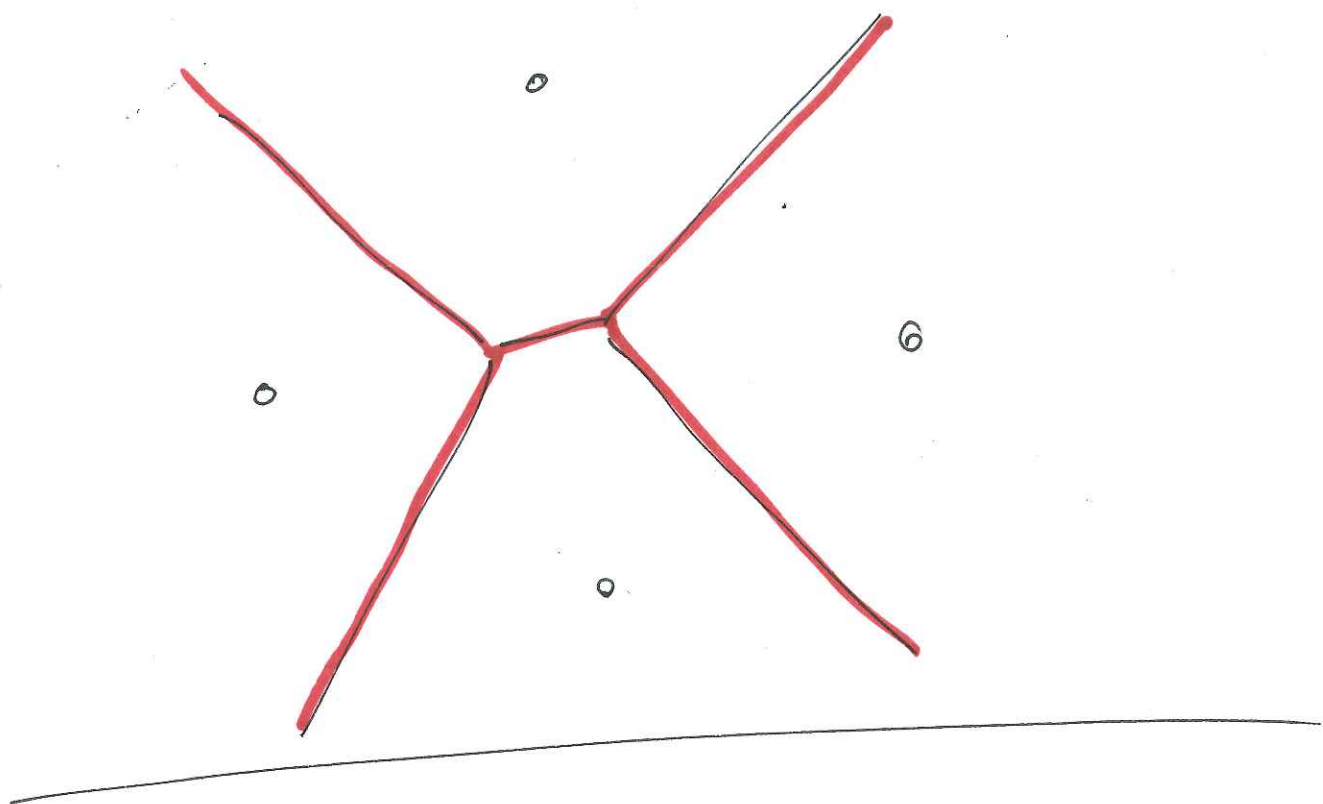


3 points



It's the
circumcenter.

4 points



Claim: V_i is a polyhedron.

pf For a fixed p_j (s.t. $p_j \neq p_i$), the ^{closed} set $\{x \in \mathbb{R}^2 : \|x - p_i\| \leq \|x - p_j\|\}$ is a halfspace H_{ij} bounded by the perpendicular bisector between p_i and p_j .

Note that dimension is irrelevant here.

$$V_i = \left\{ x \in \mathbb{R}^2 : \|x - p_i\| \leq \|x - p_j\| \quad \forall j=1 \dots n \right\}$$

$$= \bigcap_{\substack{j=1 \\ (j \neq i)}}^n \left\{ x \in \mathbb{R}^2 : \|x - p_i\| \leq \|x - p_j\| \right\}$$

$$= \bigcap_{j=1}^n H_{ij}$$

So, V_i is the intersection of a finite set of halfspaces. Therefore V_i is a polyhedron.

Cor: V_i is convex.

Claim: Vor_P is a polyhedral complex.

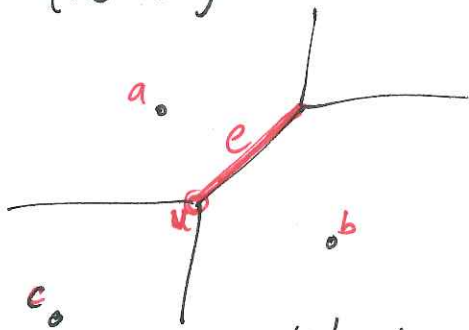
pf idea: We know each V_i is a polyhedron.

We need to show that any pair of faces intersect at a common subface.

But first, what are the faces? Polygons, Edges, Vertices

If a point is in more than one V_i then it is equidistant to more than one closest p_i .

In \mathbb{R}^2 , the edge e is the set of points whose nearest point in P is either a or b , both are the same distance.



$$\text{So, } e = V_a \cap V_b$$

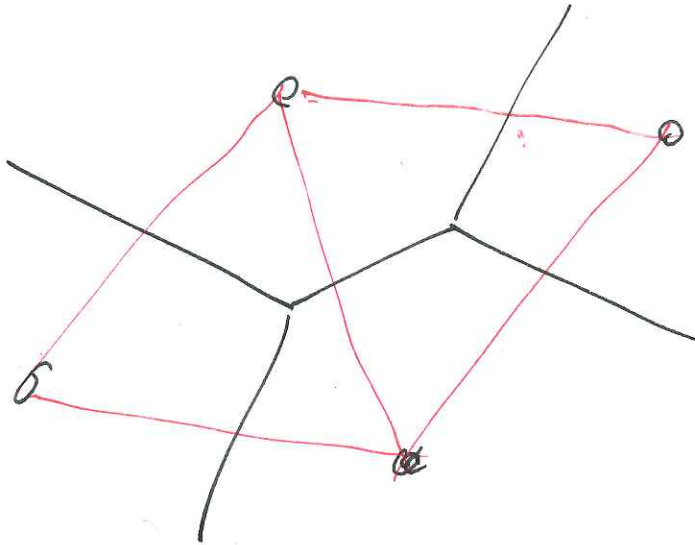
A vertex u has ^{at least} 3 nearest points in P .

$$u = V_a \cap V_b \cap V_c$$

If $x \in \bigcap_{i=1}^k V_i$ then $x \in \bigcap_{i \neq j=1}^k \text{bisector}(p_i, p_j)$

This implies x is in a face of each V_i .

The Dual of the Voronoi Diagram
is the Delaunay Triangulation.



Check, one vertex per face.
one triangle per Voronoi vertex
one edge per edge

The Voronoi Diagram is a lower envelope.

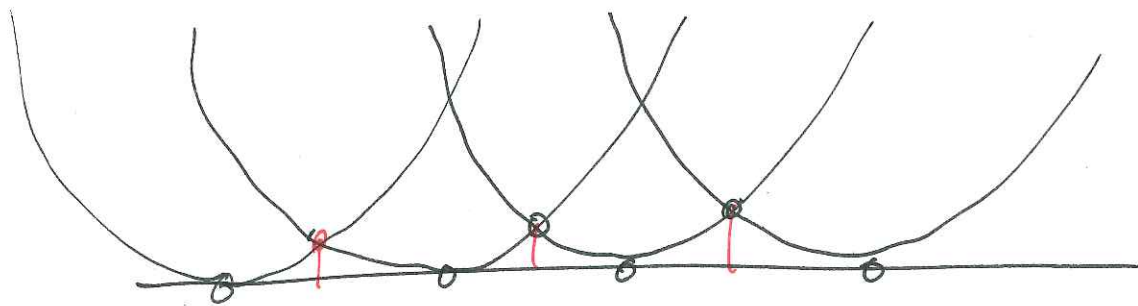
$$f(x) = \min_{i=1 \dots n} \|x - p_i\|^2$$

squared distance to p

why is this okay?

$$\|x - p_i\| \leq \|x - p_j\|$$

$$\|x - p_i\|^2 \leq \|x - p_j\|^2$$



$$\text{Let } g(x) = -\|x\|^2$$

$f+g$ is the ~~lower~~ upper hull of a polyhedron.