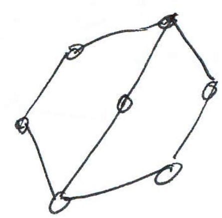
Tutte Embedding (barycentric embedding)

1) Pick some outer face and pin it down in convex position.
2) Put the remaining vertices at the centroids of nbrs.

Implementation: Requires find L_2^{-1} (minor of the Laplacian of G)

$$L_G = \left[\begin{array}{c|c} L_1 & B^T \\ \hline B & L_2 \end{array}\right] \qquad L_G P = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

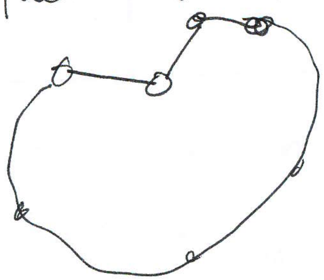
Def A **convex representation** of G is a drawing s.t. every face is a convex polygon.



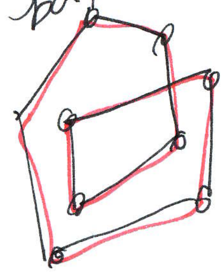
Thm [Tutte] A barycentric embedding of a planar, 3-connected graph is a convex rep.

The 4 Bad Things (that don't happen)

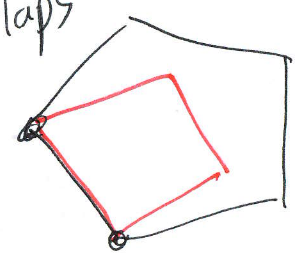
No face zig-zags



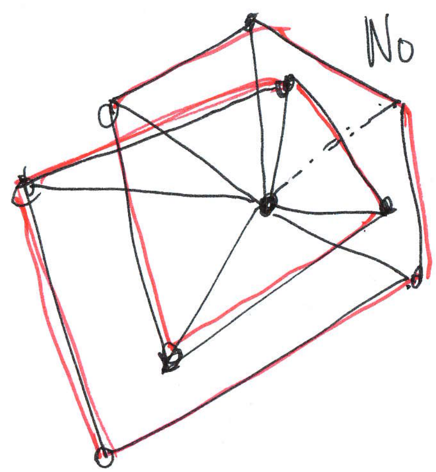
No self-crossings on face bdy.



No overlaps



No winding

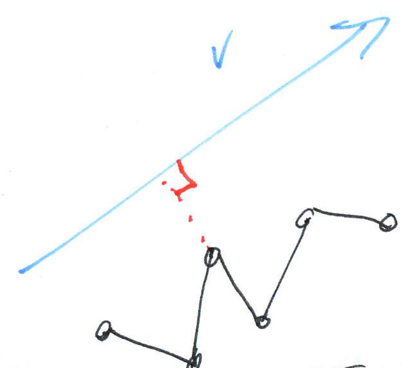


Monotone Paths

Let $v \in \mathbb{R}^2$, F = Vertices of Outer Face $p_1, \dots, p_n$

Def A path $q_0, \dots, q_k$ in G is v-monotone if

$$q_i \cdot v \geq q_{i-1} \cdot v$$

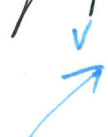


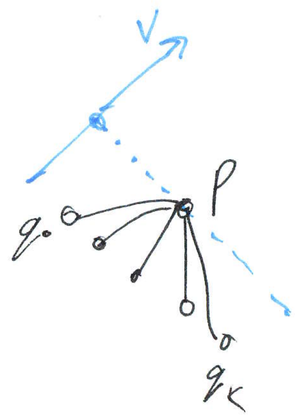
Claim: $\forall p \in V$, $v \in \mathbb{R}^2_{\neq 0}$

There exists a v-monotone path from p to F.

Pf Perturb v by some ε so that $p_i \cdot v \neq p_j \cdot v$ $\forall i,j$

If $\forall q \sim p$ have $q \cdot v < p \cdot v$ then p is not centroid.

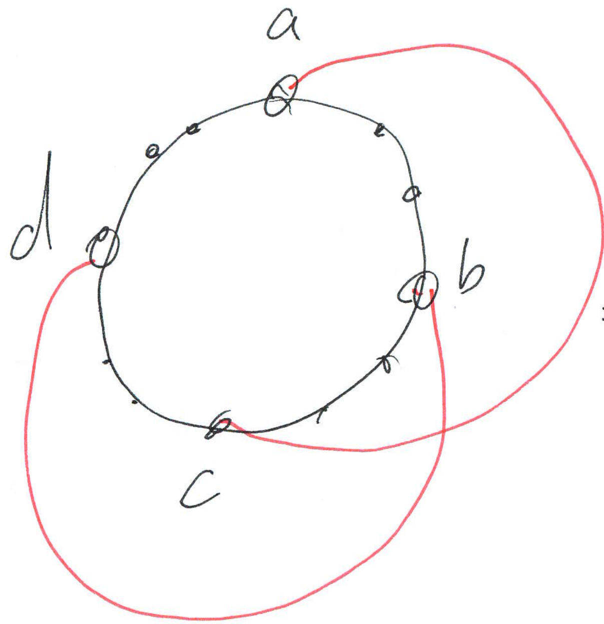




$\Rightarrow \exists q$ such that $q \cdot v > p \cdot v$

Add (p, q) to path... repeat.

Double-Crossing a face



Claim: $\nexists$ disjoint paths $a \to c$ and $b \to d$

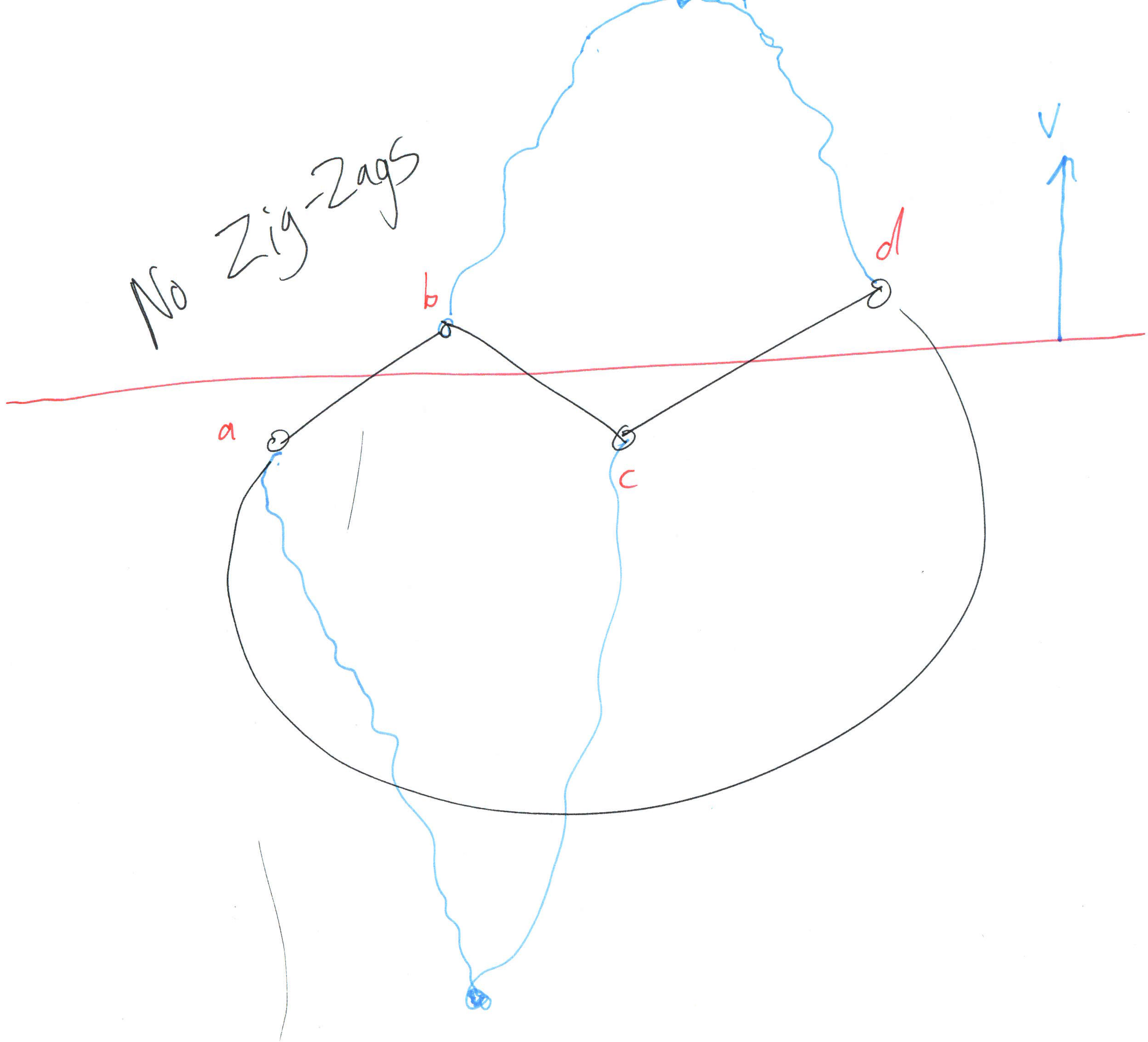
$= K_4$

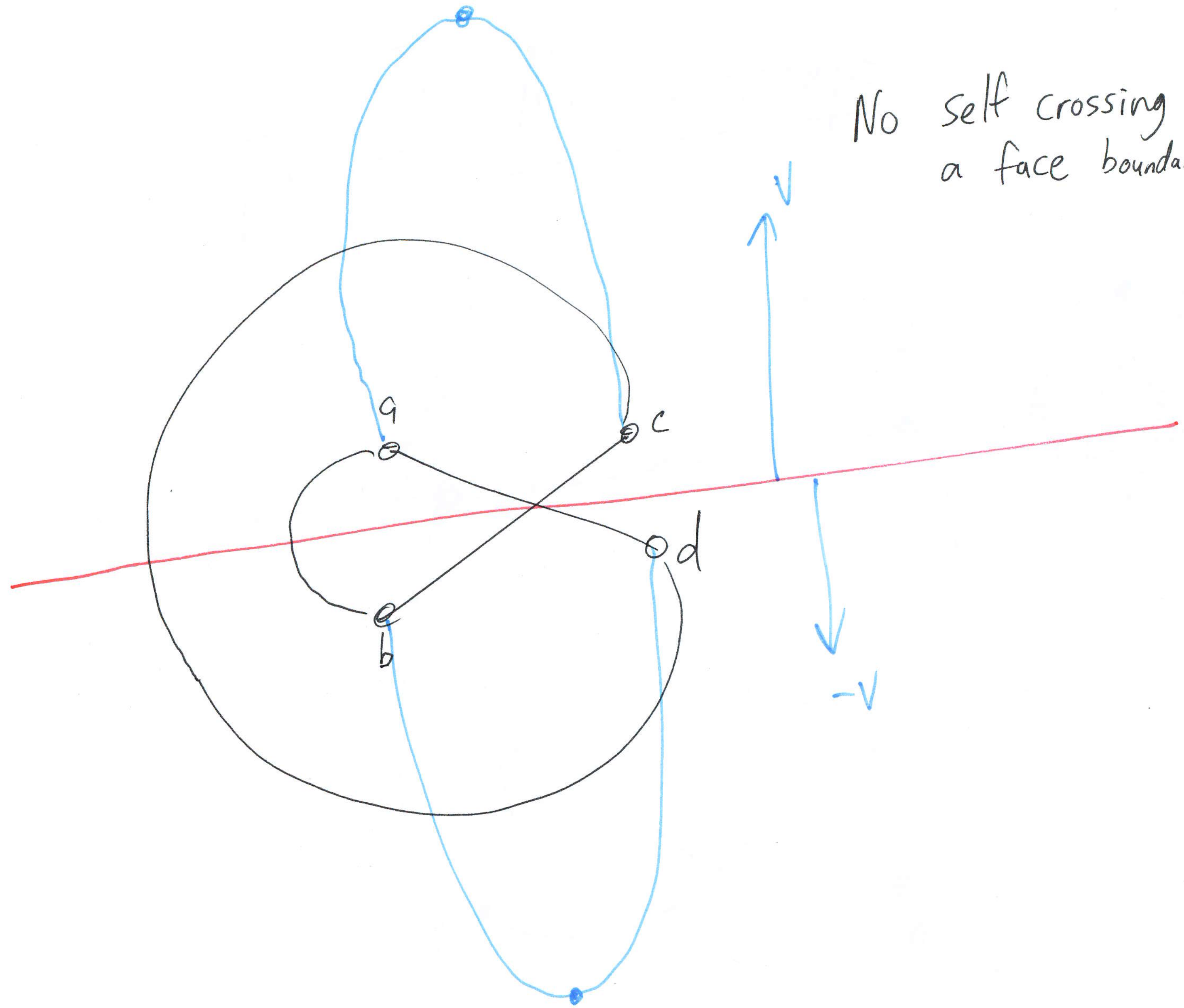
$E = 6 = 12 - 6 = 3(4) - 6$
$= 3n - 6$

$\Rightarrow$ Every face a Δ

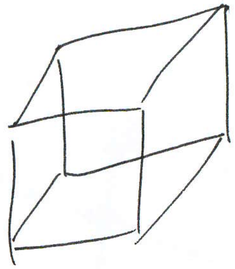
$\Rightarrow$ Contradiction.

No Zig-Zags

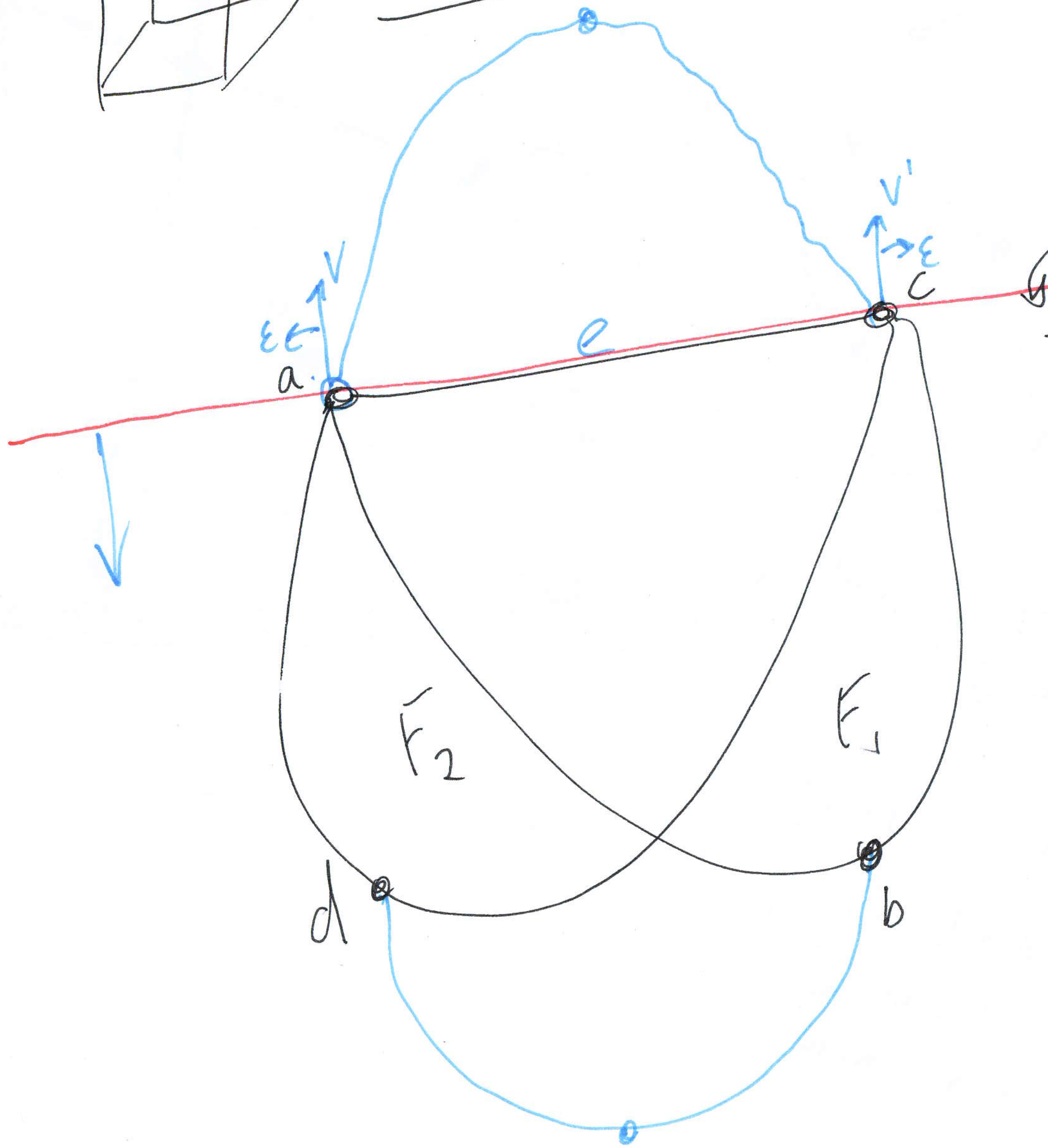


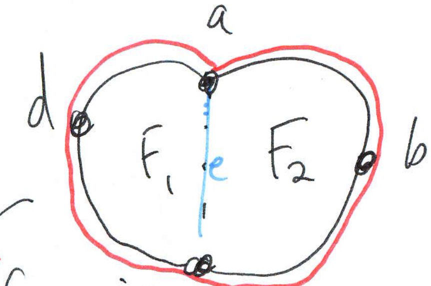


No self crossing
a face boundary

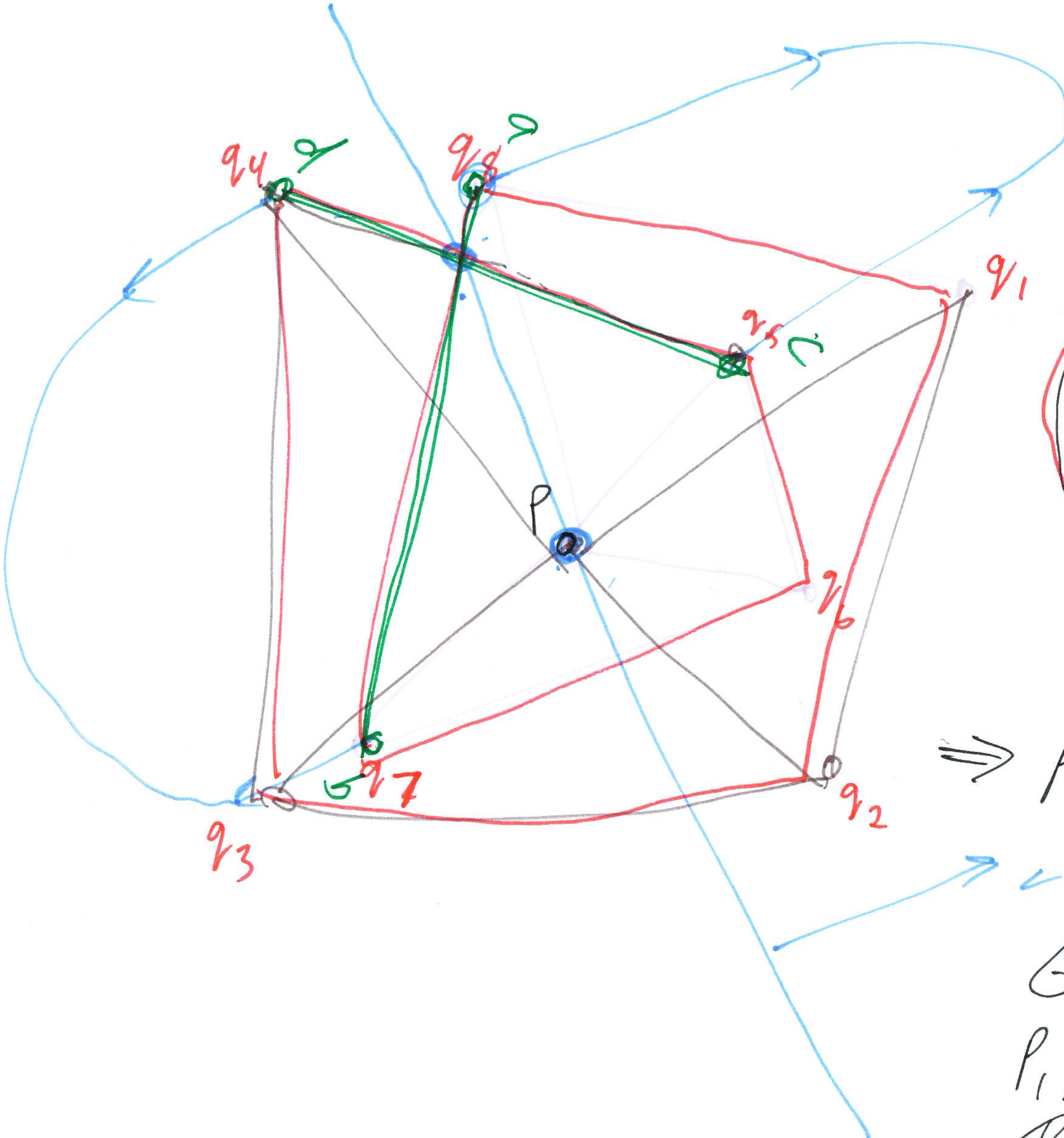


No Overlaps

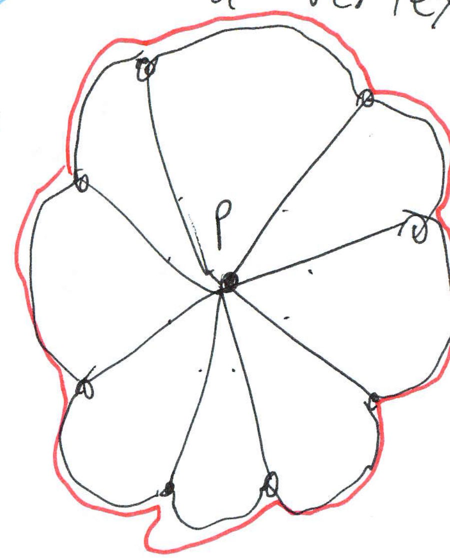




face in $G \setminus e$.



No winding around a vertex.

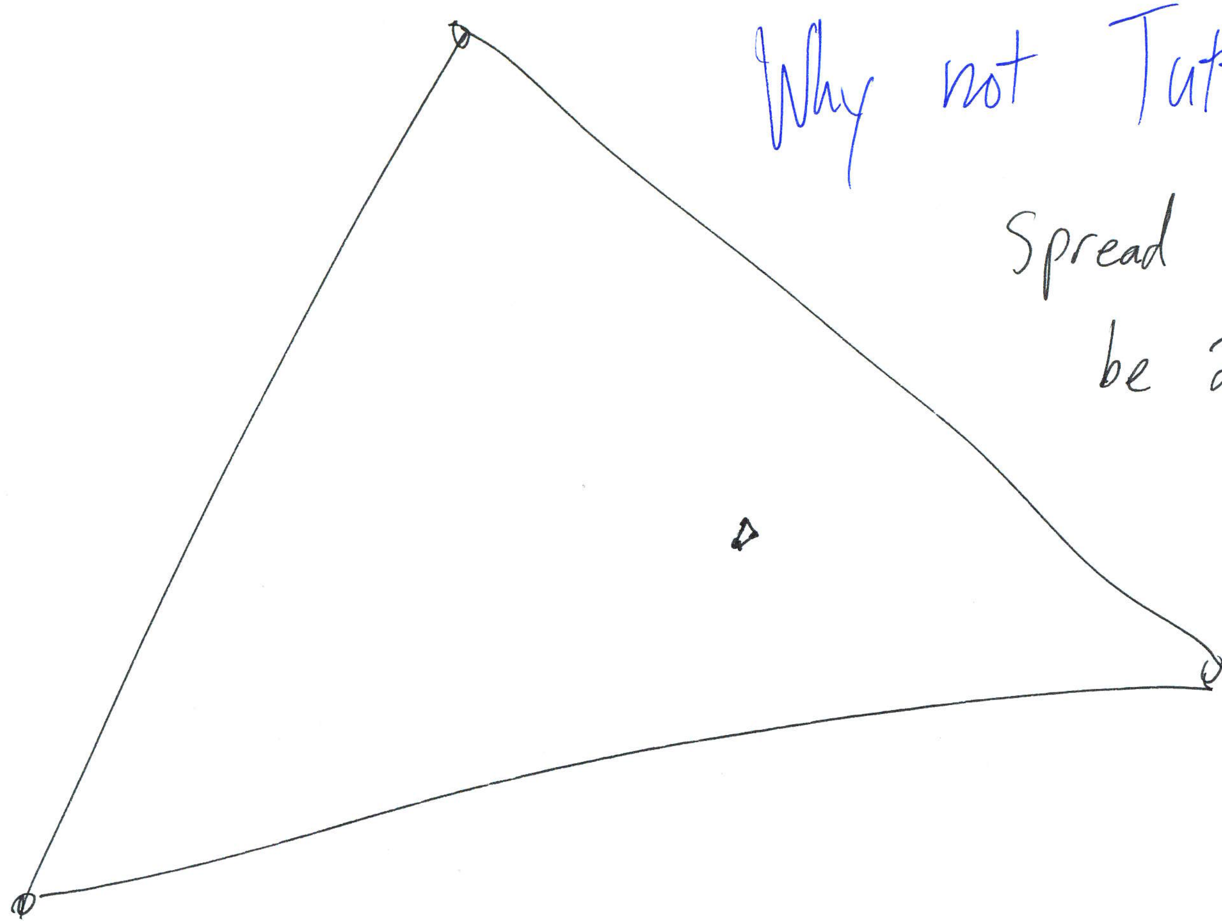


$\Rightarrow$ Paths: $P_1 : a \to c$, $P_2 : b \to d$ } disjoint

$G \setminus p$ 2-conn, planar

P_1, P_2 double-cross

The face we get from the cavity of p



Why not Tutte?

Spread could be 2^n