Computational Geometry: Lecture 14

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1 Linkages and Stresses

Today we're going to go back into history a little bit. We'll talk a bit about some great geometric ideas from the 19th century that are especially applicable to modern applications in computational geometry.

When people shifted from building bridges out of stone to building them out of metal, there was a fundamental change in the engineering involved. With stone, all the strength is compressive, so you need to see what force pushes where. With iron and steel, there is the introduction of pulling. For simplicity, we will be working in a 2-dimensional world. The systems of pushing and pulling elements is called a linkage and is defined as follows.

Definition 1.1. *A* linkage *is a graph with vertices embedded in the plane and edges represented as straight lines. The graph may not be planar and even if it is, the drawing may have crossings. When the graph is planar and the drawing has no crossings, we will call it a* planar linkage*.*

Definition 1.2. *A* stress *on a linkage is an assignment of weights to the edges of a linkage. These weights may be positive or negative and they will be interpreted as spring constants, i.e. the force exerted on the endpoints will be the weight times the length of an edge.*

Often we are interested in understanding how the stress on a linkage will move it. You may recognize a connection to Tutte's algorithm here. Recall that in Tutte's algorithm, the weights on the interior edges are all 1 and the stress pulls the graph into a nice straight-line drawing. There is a slight difference in that Tutte's algorithm didn't really account for the stresses on the outer face. We will see this difference crop up again soon.

The other type of stress we are interested in is the one where nothing moves. This is similar to the final state of Tutte's algorithm, when the linkage has reached a kind of equilibrium.

Definition 1.3. *An* equilibrium stress *on a linkage is a stress such that for each vertex, the sum of the forces induced by its incident edges is zero, assuming the weights are interpreted as spring constants.*

2 Lifting Planar Graphs and the Maxwell-Cremona Correspondence

In 1864, James Clerk Maxwell developed a theory of *reciprocal diagrams*, a special kind of planar graph dual. He showed that these reciprocal diagrams were very useful for understanding the equilibrium stresses of a planar linkage. Combined with the work of Luigi Cremona, an Italian mathematician working around the same time, we now have what is known as **the Maxwell-Cremona Correspondence**, which relates the equilibrium stresses of a planar graph to the liftings of that graph. So what's a lifting?

Definition 2.1. *A* lifting *of a planar straight-line drawing of a graph is an assignment of heights to the vertices such that the vertices of every face are coplanar in $\mathbb{R}^3$ (i.e. faces stay flat).*

The condition on faces staying flat after the lifting applies also to the outer face, a condition that will be crucial.

3 A Map for Today

We're not going to prove the Maxwell-Cremona Correspondence today. Instead, we're going to take a bit of a tour through some related ideas and cool math. On the way, we'll see some more connections to things we've already seen and set ourselves up for some other applications that will crop up later in the course.

We'll start with a continuation of our historical rummaging, by talking about a fundamental geometric problem that came out of steam engine design. This will allow us to do a kind of steampunk computational geometry, without dressing up like idiots.

Then, we connect this to a physical model of computation over the complex numbers. This will lead us to a brief discussion of stereographic maps and polarity of polytopes.

Finally, we'll formally define what we mean by a reciprocal diagram and map out the structure of the proof of the M-C Correspondence. Then, we'll go through that proof next time.

4 Making straight lines

If you're designing a steam engine, you have a simple geometric problem to solve: how to transform the linear motion of the piston to a rotational motion of the wheels. Now, you could come up with a naive solution that uses the fact that the piston is constrained to it's chamber. However, if you do that, you will quickly wear out the chamber. What you need is a linkage that will keep the piston moving in a straight-line, even if there wasn't a chamber.

James Watt came up with a linkage that comes very close to making a straight-line. It was close enough to work in a steam engine, but it had a slight wobble to its motion.

This problem got around and it was an open question as to whether such a linkage could even exist. In 1864, incidentally, the same year that Maxwell published his work on equilibrium stresses, a Frenchman names Peaucellier came up with a very simple linkage that produced a perfectly straight line from a circular motion.

Peaucellier's linkage was a big hit. According to legend, the great mathematician Sylvester presented a model of the linkage to Lord Kelvin. Kelvin was so mesmerized by the simple mechanism that he declared it "the most beautiful thing I have ever seen."

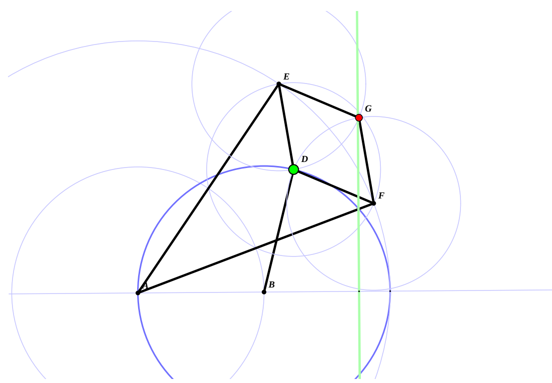

Figure 1: This is Peaucellier's linkage implemented in Cinderella.

5 Physical computation over $\mathbb{C}$

To understand the workings of the Peaucellier linkage, let's start with some even simpler linkages. We imagine that some of the vertices in the linkage are pinned down in the plane while the others can move, subject to the constraint that the lengths of the edges don't change. There are many linkages that just have one degree of freedom. That is, if we more one free vertex, the motion of the rest of the linkage is fixed. In such a linkage, we can designate one vertex as the *input* and one as the *output*. If we treat the plane as the complex numbers, linkages of this form compute a function $\mathbb{C} \to \mathbb{C}$.

Let's take a couple easy examples. For example, we can easily do addition by a constant. It is also not too hard to do multiplication by a real number, though we need a different setup for multiplying by negative numbers or numbers between 0 and 1.

Another computation we could do is called reflection around the circle. It computes the function, $f(z) = \frac{z}{|z|^2}$, where $|x + yi| = \sqrt{x^2 + y^2}$ is the *norm* of $z = x + yi$. You may also recognize this as a core part of the Peaucellier linkage.

This makes sense because reflection about the circle is used to define the *stereographic map*, which is the mathematical way of turning a circle into a line. Let C be the circle of diameter 1 centered at the point $(0, -\frac{1}{2})$ and let L be the line $y = -1$. We can define the stereographic map $s : C \setminus \{(0,0\} \to L$ as follows.

$$s(z) = \frac{z}{|z|^2}$$

This is the same function that our linkage computed but we would do well to unpack the geometric intuition a little. Suppose a point z is on C and is not the origin. We can draw a line through 0 and z that intersects L. The point where this line intersects L is exactly $s(z)$.

The part of the Peaucellier linkage beyond just reflection around the circle constrains the input vertex to stay on the circle. Thus giving us a physical computer for the stereographic map that also makes for a better steam engine.

6 Polarity

You may note that the Stereographic map is well defined in any dimension and has the same form. You may have encountered the Stereographic map before. It is often invoked to argue that the graphs formed by the vertices and edges of the platonic solids are planar. In fact, if we look at the planar graphs formed in this way, we get a natural pairing of the platonic solids by planar graph duality. The cube is dual to the octahedron, the dodecahedron is dual to the icosahedron, and the lonely tetrahedron is paired with itself.

We can also achieve this duality via a different operation known as *polarity.*

Given a (possibly infinite) set $A \subset R^d$ the polar of A, denoted A^o is defined as follows.

$$A^o = \{x \in \mathbb{R}^d : a \cdot x \leq 1, \forall a \in A\}.$$

To make this a little simpler, note that the polar of a point p is a halfspace. In particular, it is the halfspace normal to p (treated as a vector) that contains 0 and has $\frac{p}{|p|^2}$ on its boundary. Again, reflection around the circle pops up.

You could also check that the polar of a half-space is a point. Moreover, the polar of a collection of points is the intersection of halfspaces. There are many interesting facts about polarity, but for now, we'll just state that the polar of a polytope P is another polytope, not coincidentally referred to as the polar polytope of P. The polytopes P and P^o have the usual duality relationship between faces of dimension k and faces of dimension $d - k - 1$.

It is also possible to understand the dual relationship between the Delaunay triangulation and the Voronoi diagram as a kind of polarity. To do this, we would have to slightly modify our definition of polarity to do reflection around the paraboloid instead of reflection around the unit circle (sphere). It is also possible to ralize this duality using the standard polarity definition given above if you replace the parabolic lifting that we used in class with a stereographic map.

Unfortunately, we don't really have time to prove all this neat facts, but they are beautiful and I thought you should at least see them.

7 Reciprocal Diagrams and the Maxwell-Cremona Correspondence

Definition 7.1. *A* reciprocal diagram *of a drawing of a planar graph G is a linkage whose graph is dual to G and all coresponding edges are perpendicular.*

We should stop for a second and recall that the Delaunay triangulation and Voronoi diagram have this dual relationship. The Voronoi edges are on the perpendicular bisectors of the points on the corresponding Delaunay edge.

Next time we will see two important correspondences which together imply the Maxwell-Cremona correspondence.

The first is a correspondence between equilibrium stresses and reciprocal diagrams. The second is a correspondence between liftings and reciprocal diagrams. Composing these two correspondences gives us the following.

Theorem 7.1 (Maxwell-Cremona Correspondence)**.** *There is a natural correspondence between the equilibrium stresses of a planar linkage and the liftings of that linkage.*