Definitions

Straight line embedding $\rightarrow$ (Drawings are Embedding but not V.V.)

$P : V \to \mathbb{R}^2$

Stress

$s : E \to \mathbb{R}$

Force : Embedding + Stress $\rightarrow$ Force Vectors at vertices

$$\text{force at } v = \sum_{u \sim v} s_{uv}(u - v) \qquad \boxed{F = L_s P}$$

Equilibrium stress on a ~~embedd~~ SLE P is a stress $\hat{s}$ s.t. $F = 0$.

Lifting of an embedding $\quad h_0 : V \to \mathbb{R} \rightsquigarrow h : \mathbb{R}^2 \to \mathbb{R}$

linear on faces

$$h_0^+ : V \to \mathbb{R}^3 : h_0^+(v) = \begin{bmatrix} p_v \\ h_0(v) \end{bmatrix} \in \mathbb{R}^3$$

$$h_0^\star : F \to \mathbb{R}^3 \xrightarrow{\text{proj}} \text{Reciprocal dgm}$$

dual of planes through lifted faces

Recall Projective duality

$$\text{plane} \begin{bmatrix} 2a \\ 2b \\ c \end{bmatrix} \longrightarrow \text{point} \begin{bmatrix} a \\ b \\ -c \end{bmatrix}$$

$z = 2ax + 2by + c$

Face F lifts to a plane $h_F(x,y) = 2ax + 2by + c$

$$\frac{1}{2}\nabla h_F = \begin{bmatrix} a \\ b \end{bmatrix} \leftarrow \text{proj of dual to } h_F.$$

$$\boxed{\text{Lifting} \downarrow \text{Rec. Dgm}}$$

SLE for G^* given a lifting h:

$$F^* = \frac{1}{2}\nabla h_F$$

Reciprocal condition: F_1, F_2 adjacent faces in G.

~~$(p,q) \in V$ is~~ $(p,q) \in E$ is edge separating F_1, F_2.

$$(F_1^* - F_2^*)^T (p-q) = \frac{1}{2}\left((\nabla h_{F_1})^T (p-q) - (\nabla h_{F_2})^T (p-q)\right)$$

$$\left[\text{recall } (\nabla h_{F_1})^T (p-q) = \frac{h_{F_1}(p) - h_{F_1}(q)}{\|p-q\|} = \frac{h_{F_2}(p) - h_{F_2}(q)}{\|p-q\|} = \right]$$

$$= 0.$$

Lifting to Rec. Dgm

$$F^* = \frac{1}{2}\nabla h_F$$

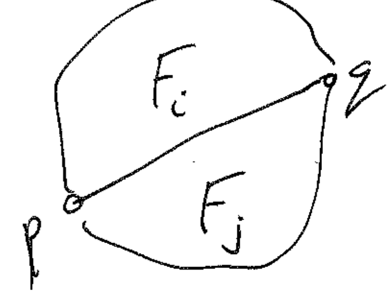
Rec. Dgm to equil. stress

Rotate rec. dgm $90°$ so dual edges are parallel.

Call the rotated dual vertices $\bar{F}_1^*, \dots, \bar{F}_m^*$.

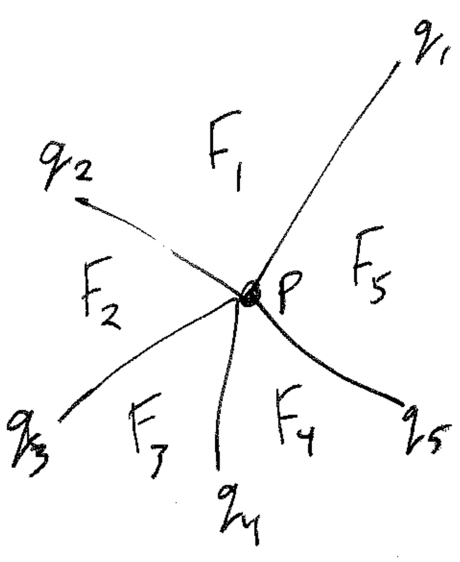
Define the stress as $\{s_{pq} : (p,q) \in E\}$ so that

Force of p on q $= s_{pq}(p-q) = \bar{F}_i^* - \bar{F}_j^*$



Check that this gives equil.

$$\sum_{q \sim p} s_{pq}(p-q) = \underbrace{\sum_{(\bar{F}_i^*, \bar{F}_j^*) \text{ dual to } (p,q)} \bar{F}_i^* - \bar{F}_j^*}_{\text{telescoping sum}} = 0$$



$$(\bar{F}_1^* - \bar{F}_2^*) + (\bar{F}_2^* - \bar{F}_3^*) + (\bar{F}_3^* - \bar{F}_4^*) + (\bar{F}_4^* - \bar{F}_5^*) + (\bar{F}_5^* - \bar{F}_1^*) = 0$$

Stress → Rec. Dgm

Fix $F_0^* = 0$, put one vertex at the origin.

For any face F_k, consider a path $(F_0^*, F_1^*, \ldots, F_k^*)$ for F_0^* to F_k^* in G^*.

Define $$F_k^* = \sum_{i=0}^{k-1} s_{e_i}(p_i - q_i)$$

where $e_i = (p_i, q_i)$ is the edge of G dual to (F_i^*, F_{i+1}^*).

$\boxed{\text{\underline{Fact}: Choice of path doesn't matter.}}$

Check $F_{i+1}^* - F_i^* = s_{e_i}(p_i - q_i)$.

So, edges (F_i, F_{i+1}) and (p_i, q_i) are parallel.

Rotate everything 90° (if it matters).

Rec

Dgm $\rightarrow$ Lifting

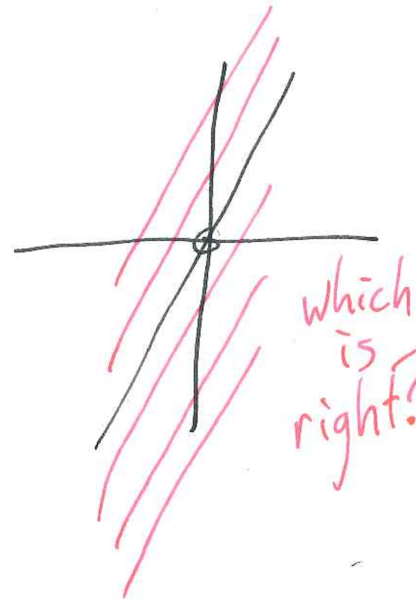
For each face F, we have a point $F^* = \nabla h_F$.

We want to go from $(\nabla h_{F_1}, \dots, \nabla h_{F_m})$ to h.

It's integration! Piecewise constant functions.

$$\int 3\,dx = 3x \boxed{+\, c}$$

Don't Forget!



Fix F_0 and set $c_0 = 0$, i.e. $h_{F_0}(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}^T F_0^*$

All other faces $h_{F_i}(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}^T F_i^* + \underbrace{c_i}_{\text{Need to find } c_i\text{'s}}$

For any F_k consider the path $(F_0^*, \dots, F_k^*)$ in G^*.

Set c_i s.t. $h_{F_i}(p_i) = h_{F_{i+1}}(p_i)$

Fact: Choice of path doesn't matter.