Delaunay Δ^n:

(1) Lift points:
(2) Project Lower Hull

$$\begin{tikzcd} \text{Points} \arrow[r] \arrow[d, dashed] & \text{Graph} \\ \text{Lifting} \arrow[ur] & \end{tikzcd}$$

Tuttes Algorithm

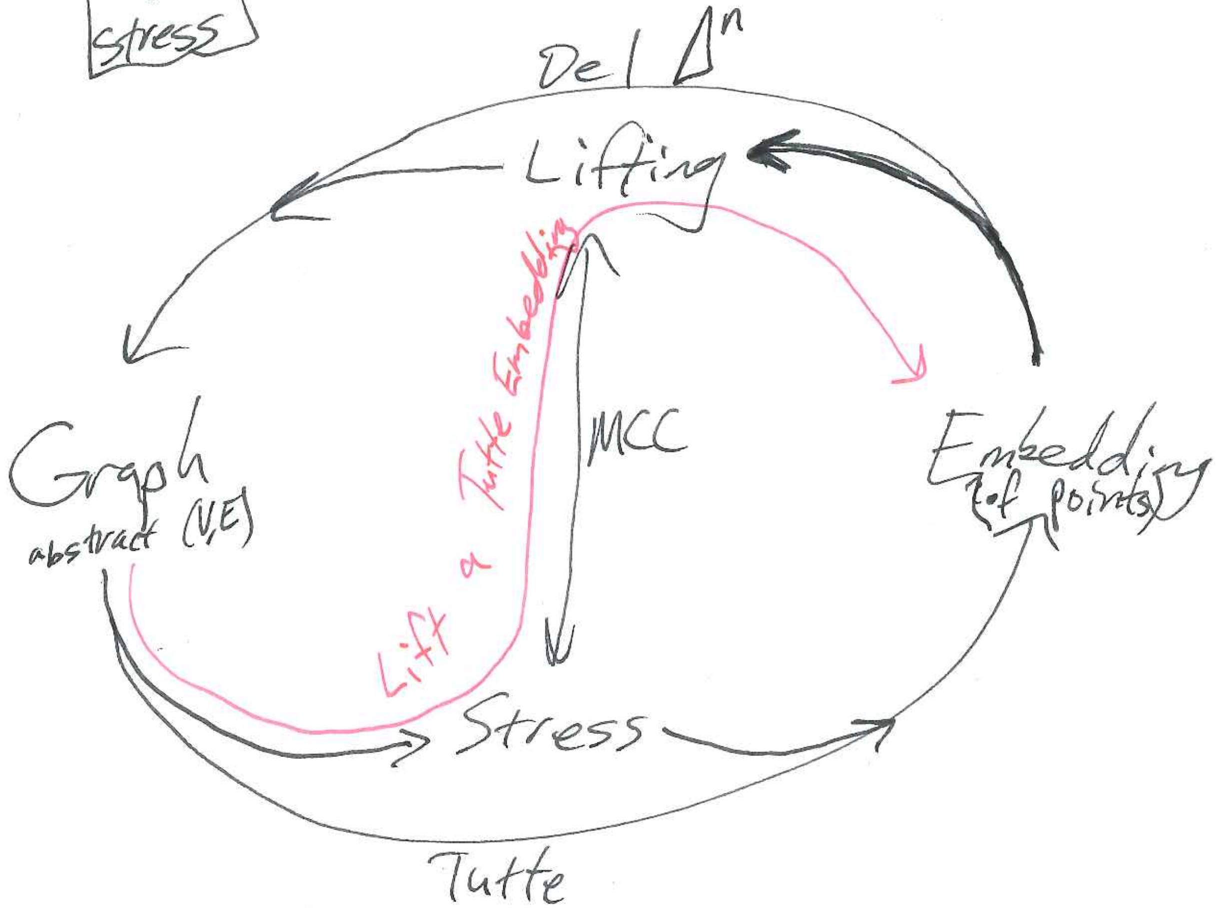
(1) Graph is input
~~(2) Find equil~~ ~~partial~~
(2) Pick a drawing to balance ~~stress~~ forces

$$\begin{tikzcd} \text{Graph} \arrow[r] \arrow[d] & \text{points} \\ \text{stress} \arrow[ur] & \end{tikzcd}$$

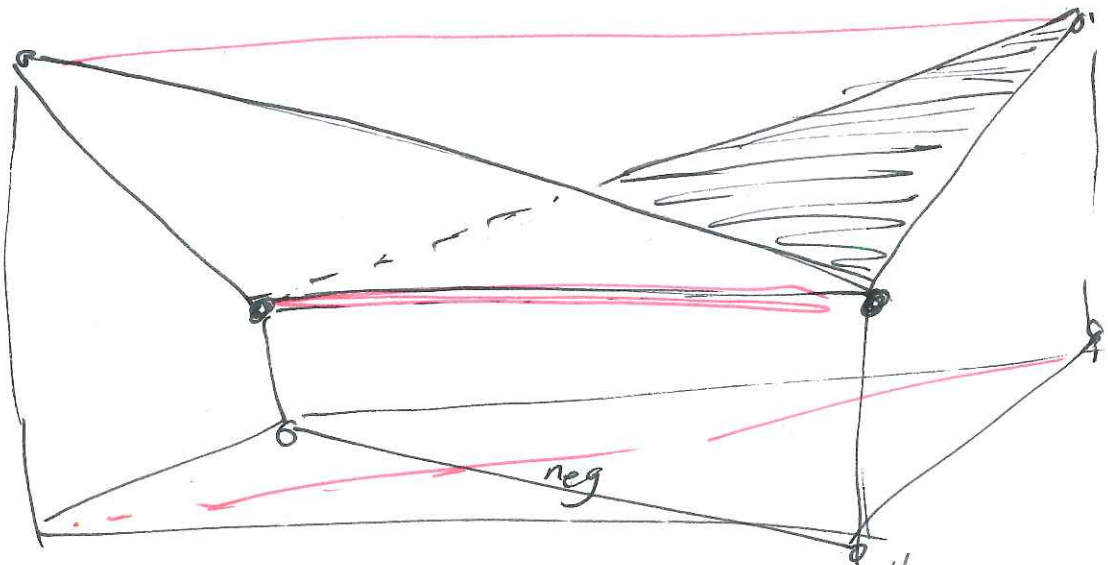
MCC

(1) Graph is input with drawing
(2)(a) Lift and get eq. stress
(2)(b) Choose eq. stress + get lifting

$$\text{Graph} + \text{points} \rightarrow \boxed{\text{lifting} \Updownarrow \text{stress}}$$



Sign of the stress related to local Del. cond.



neg stress $\Rightarrow$ valley

pos. stress $\Rightarrow$ mountain

Weighted Delaunay Δ^n

For each $p \in P$ pick $w_p \in \mathbb{R}$

Lift $p \longmapsto \underbrace{\begin{bmatrix} p \\ \|p\|^2 - w_p^2 \end{bmatrix}}_{\|p\|^2 - w_p^2}$

Note $w_p = 0$ gives the usual parabolic lift.

Now, project the lower hull of the lifted points.

Result is the <u>Weighted Delaunay Δ^n</u>.

Is there a dual <u>weighted Voronoi Diagram?</u>

Yes. As before, it's the projection of the upper envelope of the planes dual to the lifted pts.

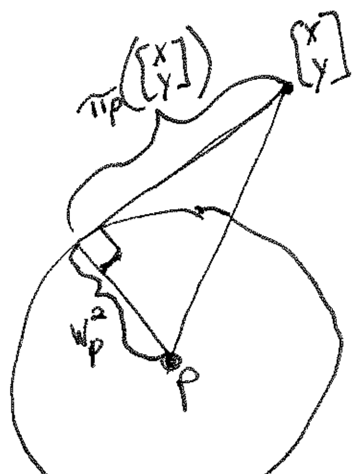
point + weight → (p, w_p) ⟷ lifted point in $\mathbb{R}^3$ $\begin{bmatrix} p \\ \|p\|^2 - w_p^2 \end{bmatrix}$ ⟶ (non-vertical) plane in $\mathbb{R}^3$ $\left\{ z = 2p^T\begin{bmatrix} x \\ y \end{bmatrix} + w_p^2 - \|p\|^2 \right\}$

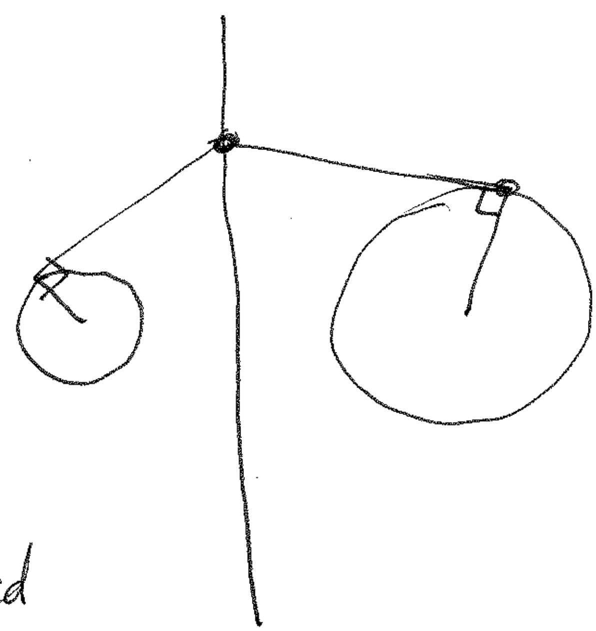
What is the "bisector" between (p, w_p) and (q, w_q)?

$$\|p\|^2 - 2p^T\begin{bmatrix} x \\ y \end{bmatrix} - w_p^2 = \|q\|^2 - 2q^T\begin{bmatrix} x \\ y \end{bmatrix} - w_q^2$$

note: $\left\|p - \begin{bmatrix} x \\ y \end{bmatrix}\right\|^2 = \|p\|^2 - 2p^T\begin{bmatrix} x \\ y \end{bmatrix} + \left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\|^2$

$$\underbrace{\left\|p - \begin{bmatrix} x \\ y \end{bmatrix}\right\|^2 - w_p^2}_{\text{call this } \pi_p\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)} = \underbrace{\left\|q - \begin{bmatrix} x \\ y \end{bmatrix}\right\|^2 - w_q^2}_{\text{call this } \pi_q\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)}$$





π_p is sometimes called the <u>power distance</u>.

Recall: Voronoi cells: $\mathrm{Vor}_P(q) = \{x \in \mathbb{R}^2 : d(x,q) \le d(x,y) \ \forall y \in P\}$

Now: P': points with weights.

Weighted Voronoi cells: $\mathrm{Vor}_{P'}(q) = \{x \in \mathbb{R}^2 : \pi_q(x) \le \pi_y(x) \ \forall y \in P'\}$

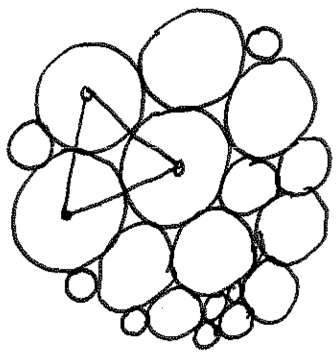
<u>Fact</u>: The Tutte Embedding is a weighted Delaunay "diagram".

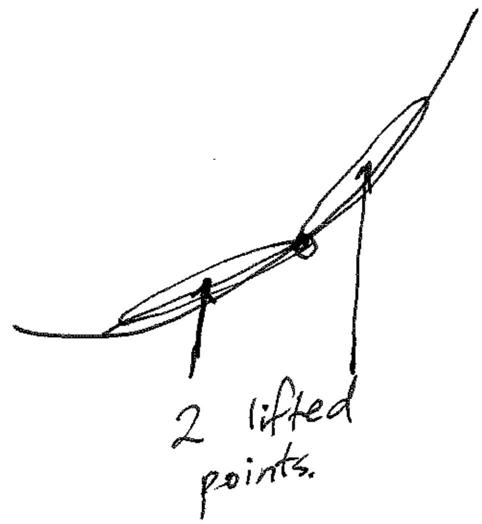
What if it isn't a triangulation?

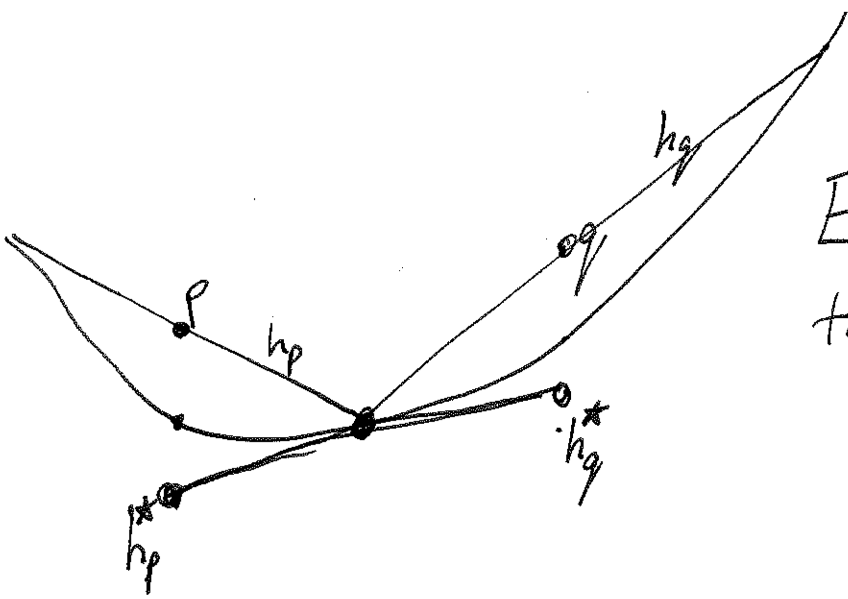
<u>If</u> some facet of the lower hull is not a triangle, we will get non-triangular faces. (recall MC lifting condition)

$\Rightarrow$ It is a projection of a lower hull.

Koebe Embeddings are weighted Delaunay Δ's.







Edges are tangent to the paraboloid.

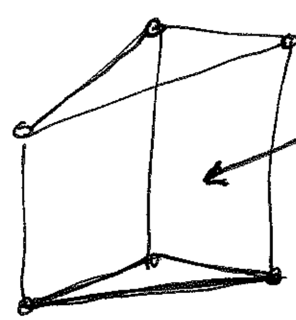
From a Triangulation to a function on the vertices

For each vertex v let a_v = area of the triangles around v.

If $h: V \to \mathbb{R}$ is a height fn, we can treat it as a vector in $\mathbb{R}^n$ (i.e. $h_i := h(v_i)$)

not nec. convex

Note: Volume beneath the lifted Δ^n is $\frac{1}{3} h^T v$.



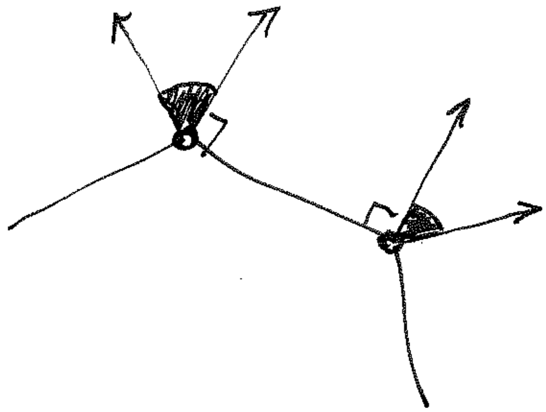
Volume is area of base time average height.

So, the lower hull minimizes the volume among all Δ^ns of a lifted point set. It's a linear ~~pg~~ program!

The secondary polytope of a point set is the convex ~~hu~~ closure of the area vectors of all triangulations of the point set.

The vertices of the sec. polytope are weighted Del Δ^{n}_{s}

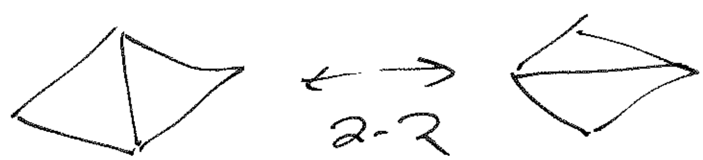
The edges of the sec. polytope are flips!



Directions are height functions.

The objective function is the volume below.

Generically, two triangulations with the same area below have only 4 coplanar points



Note:
Some points disappear.