Euler's Formula for the plane:

$$\chi = |V| - |E| + |F| = 2$$

Gauss-Bonnet $\sum_{v \in V} K(v) = 2\pi\chi$,

where $K(v) = 2\pi - \sum \text{angles at } v$ and

$$\chi = |V| - |E| + |F|$$

pf

$$\sum_{v \in V} K(v) = \sum_{v \in V} \left(2\pi - \sum_{\substack{f \text{ incident} \\ \text{to } v}} (\text{angle of } f \text{ at } v)\right)$$

$$= 2\pi|V| - \sum_{v \in V} \sum_{f \sim v} \text{angle of } f \text{ at } v$$

$$= 2\pi|V| - \sum_{f \in F} \sum_{v \sim f} \text{angle of } f \text{ at } v$$

$$= 2\pi|V| - \sum_{f \in F} (|f| - 2)\pi$$

$$= 2\pi(|V| - |E| + |F|) \quad \text{because } \sum_{f \in F} |f| = 2|E|$$

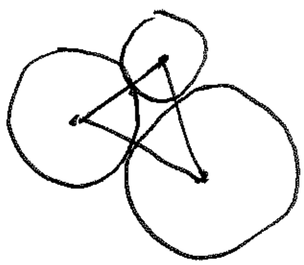
Koebe Embedding

Input: Planar $G=(V,E)$
Output: Circle Packing Embedding of G

Let's only consider Δ^ns.
(if it's not a Δ^n, triangulate the graph by adding a vertex in each face. Later, remove these vertices.)

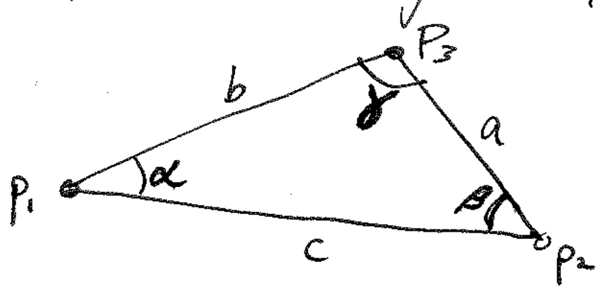
Claim: It suffices to find the radii.

Idea:



3 radii determine the triangle (up to congruence)

Given the radii r_1, r_2, and r_3, we can find the angles by the Law of Cosines



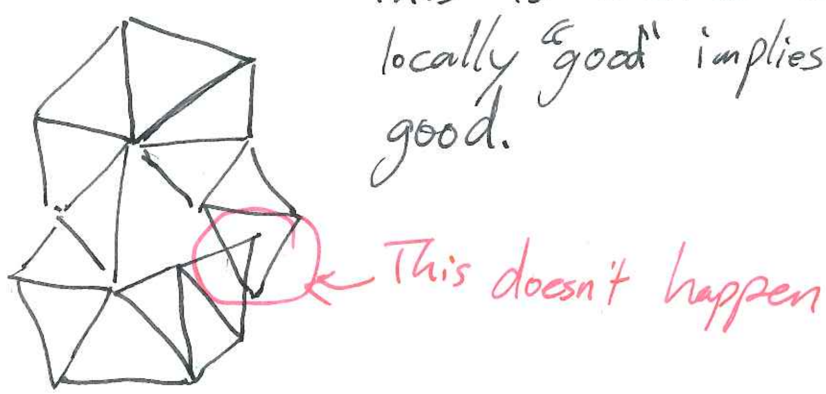
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

$$c = r_1 + r_2$$
$$b = r_1 + r_3$$
$$a = r_2 + r_3$$

Imagine this: $r = (r_1, \ldots, r_n)$ is an assignment of radii.

For each triangle face $\{v_i, v_j, v_k\} \in F$, we cut the corresponding triangle out of wood.

Claim: If we glue the triangles edge to edge, as long as the angles add up to 2π at each vertex, we will get a nice embedding



This is another case of locally "good" implies globally good.

Connection to Curvature: $\sum_{\text{angles at } v} = 2\pi \Rightarrow \underbrace{K(v) = 0}_{\text{Flatness.}}$

Not flat at outer face

Fix outer face: $\sum_{\text{angles at } v} = \frac{2\pi}{3}$ for v outer.

Let $S = \{ r : \sum r_i = 1 \text{ and } r_i \geq 0 \}$

Let $H = \underbrace{\{ x \mid \sum x_i = (2n-4)\pi \}}_{\text{flat embeddings}}$

$(n-3)2\pi + 3\left(\frac{2\pi}{3}\right) = (2n-4)\pi$

$f : S \to H$ by $f(r) = (\sigma_r(v_1), \ldots, \sigma_r(v_n))$

$\sigma_r(v) = \sum(\text{angles around } v)$

<u>Fact</u>: f is a bijection into the ~~subspace~~ subset P^* of H

defined as $\sum x_i = (2n-4)\pi$ ← globally flat

$\forall I \subset [n] \quad \sum_{i \in I} x_i < |F(I)|\pi$ ← curvature ~~positive~~ nonneg. locally.

\# of faces incident to v_i for $i \in I$.

Existence of a Koebe embedding $x^* = (2\pi/3, 2\pi/3, 2\pi/3, 2\pi, 2\pi, \ldots, 2\pi)$

Check $|F(I)| > 2|I| \quad \forall I \subset [n] \quad |I| \leq n-3$

$\sum_{i \in I} x_i^* \leq 2\pi |I| < |F(I)|\pi \Rightarrow$ done

If $|I| = n-1$ or $|I| = n-2$ the $F(I) = F \Rightarrow$ easy

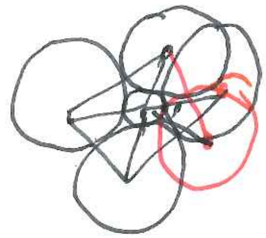
An Algorithm

Search for r

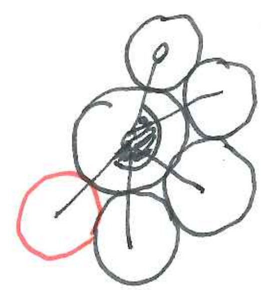
Start with a guess $r = (\frac{1}{n}, \dots, \frac{1}{n})$

Adjust radii to reduce curvature (towards 0) at each vertex.

Observe $\sigma_r(v_i) > 2\pi \Rightarrow$ need to increas r_i



$\sigma_r(v_i) < 2\pi \Rightarrow$ need to decrease r_i



Find r_*' s.t. if $r_j = r_*'$ $\forall v_j \sim v_i$ then $\sigma_r(v_i) = \sigma_{r'}(v_i)$

Set r_i'' s.t $\sigma_{r''}(v_i) = 2\pi$.

[r' is r with nbrs of v_i set to r_*']

[r'' is r' with updated r_i'']