Computational Geometry in Review

Perspective: Linear algebra as a tool for "seeing" high dimensional spaces.

Outline

Predicates

Convex Hulls

PSLGs → Polyhedral Complexes

Delaunay + Voronoi

Projective Duality

Other Dualities
- LP Duality
- other proj. duals
- Vector Space Duality

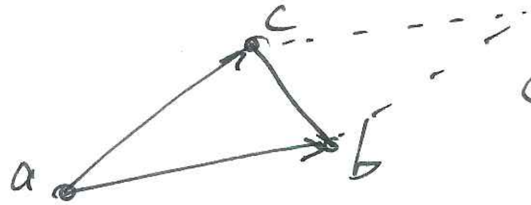
Planar Graphs

- Tutte
- Maxwell Cremona
- Searching Planar Subdivisions
- Halfspace Range Counting

why?
- Euler's Formula
- Faces generate all cycles
- Local to Global properties
- "Visible" liftings

Predicates

Simple questions to generalize comparison.

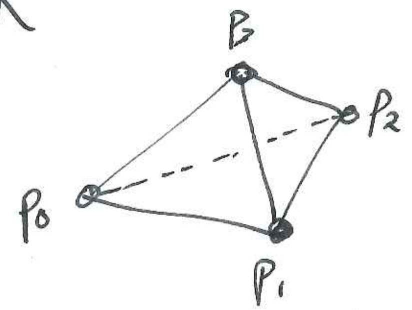


CW or CCW or collinear

$$\operatorname{sign}\left(\det\begin{bmatrix} b-a & c-a \end{bmatrix}\right) = \operatorname{sign}\left(\det\begin{bmatrix} a & b & c \\ 1 & 1 & 1 \end{bmatrix}\right)$$

homogeneous coords.

In $\mathbb{R}^3$



Is the simplex oriented inwards or outwards?

It's the same computation in $\mathbb{R}^d$

$$\underbrace{\operatorname{sign}\left(\det\begin{bmatrix} p_0 & \cdots & p_d \\ 1 & \cdots & 1 \end{bmatrix}\right)}_{\text{orientation of a } d\text{-simplex.}}$$

In~~Circle~~sphere Predicates

Is x in the sphere circumscribing $p_0, \dots, p_d \in \mathbb{R}^d$

$$\frac{\operatorname{sign}\left(\det\begin{bmatrix} x & p_0 & \cdots & p_d \\ \|x\|^2 & \|p_0\|^2 & \cdots & \|p_d\|^2 \\ 1 & 1 & \cdots & 1 \end{bmatrix}\right)}{\operatorname{sign}\left(\det\begin{bmatrix} p_0 & \cdots & p_d \\ 1 & \cdots & 1 \end{bmatrix}\right)}$$

Say $\bar{p_i} = \begin{bmatrix} p_i \\ \|p_i\|^2 \end{bmatrix} \in \mathbb{R}^{d+1}$ $\quad \bar{h} = \underset{\text{hyper}}{\wedge}\text{plane}(\bar{p_0}, \dots, \bar{p_d})$

$$\bar{h}^* = \begin{bmatrix} h \\ h_d \end{bmatrix} \qquad \bar{p_i} \in \bar{h} \Rightarrow \|p_i\|^2 = 2p_i^T h - h_d$$

$$\forall i \quad \|p_i - h\|^2 = \|p_i\|^2 - 2p_i^T h + \|h\|^2$$

$$= -h_d + \|h\|^2$$

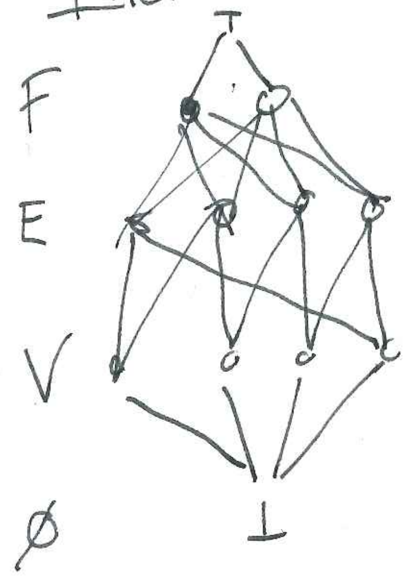
$\Rightarrow$ h is equidistant from each p_i

So, dual of $\bar{h}$ encodes circumcenter h

and radius $\sqrt{\|h\|^2 - h_d}$

PSLG Data Structures

Incidence Structure $\Rightarrow$ Poset



Combinatorial Dual.

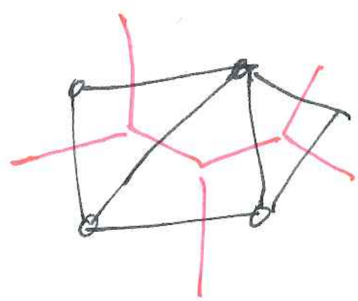
$F \leftrightarrow V'$

$E \leftrightarrow E'$

$V \leftrightarrow F'$

Convexity and Projective Duality Make the Combinatorial Duality Geometric!

Best Example

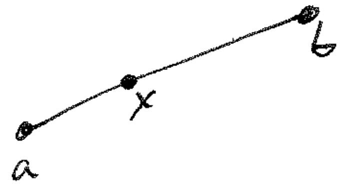


Delaunay Triangulation $\longleftrightarrow$ Voronoi Diagrams

Convex Hulls + PSLGs

Convex Combinations

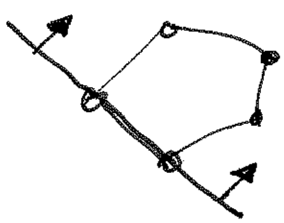
In 1D: line segments



$$\{x = (1-t)a + tb : t \in [0,1]\}$$

In General: $\sum \alpha_i p_i$; $\sum \alpha_i = 1$; $\forall i: \alpha_i \geq 0$

Support planes $\Longrightarrow$ Supporting hyperplanes



$\Downarrow$

Polyhedra

$\Downarrow$

PSLGs $\Longrightarrow$ Polyhedral Complexes

- Delaunay
- Voronoi
- Tutte Embeddings

Lifting Planar Graphs

$$
\begin{tikzcd}
 & \underset{\text{lower convex hull}}{\mathbb{RP}^3} \arrow[r, "\text{dual}"] & \underset{\text{upper envelope}}{\mathbb{RP}^3} \arrow[dr, "\text{Project}"] & \\
\mathbb{R}^2 \arrow[ur, "\text{lift}"] \arrow[rrr, dashed] & & & \mathbb{R}^2
\end{tikzcd}
$$

Delaunay/Voronoi

Tutte

Maxwell-Cremona Correspondence

Koebe Embeddings