# Tighter Bounds on the Size of Optimal Meshes

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## Abstract

The theory of optimal size meshes gives a method for analyzing the output size of a Delaunay refinement mesh in terms of the integral of a sizing function over the space. The input points define a maximal such sizing function called the feature size. Integrating the feature size function over input domain is not easy, and historically, was not deemed necessary. Matching upper and lower bounds in terms of this integral seemed sufficient. However, a new analysis of the feature size integral [9] led to linear-size Delaunay meshes [9], the Scaffold Theorem relating surface and volume meshes [8], and a time-optimal output-sensitive point meshing algorithm [10]. The key idea is to consider the pacing of an ordered point set, a measure of the rate of change in the feature size as points are added one at a time. In previous work, Miller et al. showed that if an ordered point set has pacing $\phi$, then the number of vertices in an optimal mesh will be $O(\phi^d n)$, where $d$ is the input dimension. We give a new analysis of this integral showing that the output size is only $O(n \log \phi)$. The new analysis tightens all of the previous results mentioned above and provides matching lower bounds.


## 1 Introduction

Delaunay refinement is the adding of new vertices to an initial starting set, so that the Delaunay simplices all have a bounded circumradius-to-shortest-edge ratio (see Figure 1). A celebrated result of Ruppert [13] showed how to construct such meshes with an optimal number of vertices. The straightforward generalization of Ruppert's work to $\mathbb{R}^d$ says that the number of vertices in an optimal mesh of a domain $\Omega \subset \mathbb{R}^d$ starting with a point set $P$ is bounded by the **feature size integral**:

$$\Theta\left(\int_\Omega \frac{dx}{\mathbf{f}_P(x)^d}.\right)$$

The function $\mathbf{f}_P : \mathbb{R}_d \to \mathbb{R}_{\geq 0}$ is the **feature size** induced by the input set $P$ and is defined as

$$\mathbf{f}_P(x) = \min\{r : |P \cap \mathbf{ball}(x, r)| \geq 2\}.$$

Clearly, $\mathbf{f}_P$ is 1-Lipschitz. Here and throughout, the asymptotic bounds suppress constants that are singly exponential in $d$.

Such high dimensional meshes are ideal for geometric and topological inference as they provide a nice basis for a space of smooth functions graded according to the density of the input points [7, 14]. Such functions, such as the distance function to the input or the distance to the empirical measure, can be used to recover the homology of the underlying space from which the input was sampled [3, 2].

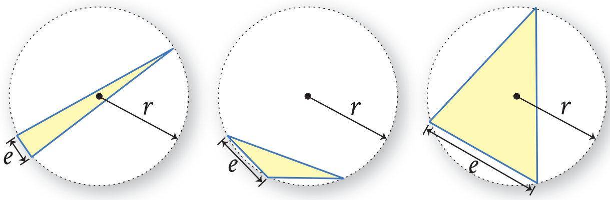


Figure 1: Three examples of triangles with different circumradius to shortest edge ratios. Delaunay refinement attempts to produce a mesh in which all simplices have this ratio bounded by a constant.

The tight bounds on mesh sizing from the Ruppert bounds are nice, but they are not very informative because they depend on the feature size integral. They do not, for example, tell when the output size will be $O(n^2)$ or $O(n)$ or any other tidy function of $n$. Moreover, they do not give an easy way to evaluate the amortized change in the output size as a result of adding a single new point.

In this paper, we give a new analysis of the feature size integral that provides tight upper and lower bounds. Our analysis makes it clear exactly when an input will yield an optimal mesh of $O(n)$ size. Moreover, the analysis, gives a tight bound on the influence of a single new point, which may have implications for future algorithms, particularly in dynamic meshing.

If we order the input set $P = \{p_1, \ldots, p_n\}$, we can define the $i$th **prefix** of the ordering to be $P_i = \{p_1, \ldots, p_i\}$. For any point $p_i$ in the ordering ($i \geq 3$), the **pacing** is defined as the ratio $\phi_i$ of the feature sizes at $p_i$ induced by $P_{i-1}$ and $P_i$:

$$\phi_i = \frac{\mathbf{f}_{P_{i-1}(p_i)}}{\mathbf{f}_{P_i(p_i)}}.$$

Let $\phi_P = \max_i \phi_i$ denote the **pacing of the ordering**.

In previous work, it was shown that the feature size integral is at most $O(n\phi_P^d)$ [9]. Theorem 1 (below) implies a bound of $O(n \log \phi_P)$, eliminating the exponential dependence on $d$. This tightens the previous results using this method, moving them from theoretically novel to practically useful.

We need a few definitions in order to state the main result. We will use $|\cdot|$ to denote the Euclidean norm for points and cardinality for sets. Let $\mathbf{ball}(c, r)$ denote the closed Euclidean ball of radius $r$ centered at $c$. Let $\mathbb{V}_d$ denote the volume of the unit ball $\mathcal{B} = \mathbf{ball}(0, 1)$, so $d\mathbb{V}_d$ is the $(d-1)$-dimensional volume of the sphere bounding $\mathcal{B}$.

**Theorem 1** *Let $P = \{p_1, \ldots, p_n\}$ be a set of $n$ points in $\mathbb{R}^d$ such that $p_1$ and $p_2$ are the farthest pair. Let $\Omega$ be a subset of $\mathbb{R}^d$ such that*

$$\mathbf{ball}(p_1, 2|p_1 - p_2|) \subseteq \Omega \subseteq \mathbf{ball}(p_1, c|p_1 - p_2|),$$

*for some constant $c \geq 2$. Then,*

$$\int_\Omega \frac{dx}{\mathbf{f}_P(x)^d} = \mathbb{V}_d \Theta \left( n + \sum_{i=3}^{n} \log \phi_i \right).$$

The proof will be broken up into two parts: the upper bound in Theorem 4 and the lower bound in Theorem 6.[1]

## 2 Related Work

The spread $\Delta_P$ of a point set $P$ is the ratio of the largest to smallest interpoint distances. A standard bad case for meshing illustrated on the left in Figure 2 occurs when there is high spread and a large empty annulus around the close points. On the right of Figure 2, the point set has been refined. Approximately a constant number of points appear in each of the geometrically growing annuli. Thus the number of points added is roughly the log of the ratio of the inner and outer radius. Observe that other than the inner most annuli, the number of points added is not heavily affected by the number of points on the inside. As a result, previous methods to characterize the feature size locally such as the gap ratio introduced by Talmor [15] can dramatically overestimate the feature size integral because each interior point pays for all of the refinement. In the worst case, such an analyses lead to an $O(n \log \Delta_P)$ upper bound when the true answer is $\Theta(n)$.

Erickson also used the spread to bound the complexity of Delaunay triangulations in the absence of refinement [5, 6]. However, even point sets with exponentially large spread can yield linear size meshes. Figure 3 illustrates a point set with geometrically growing spread, but the pacing is small, thus the mesh size will be linear.

[1]The asymptotic version of the lower bound as stated in Theorem 1 also depends on the trivial $\Omega(n)$ lower bound from the Ruppert bounds.

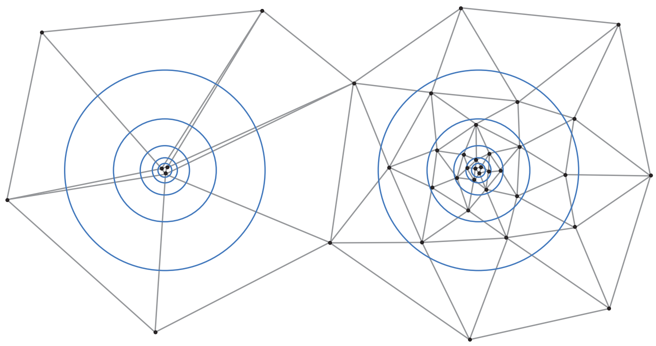

Figure 2: On the left is a typical bad example for meshing, a small set of points inside a large empty annulus. On the right shows a similar example after refinement.

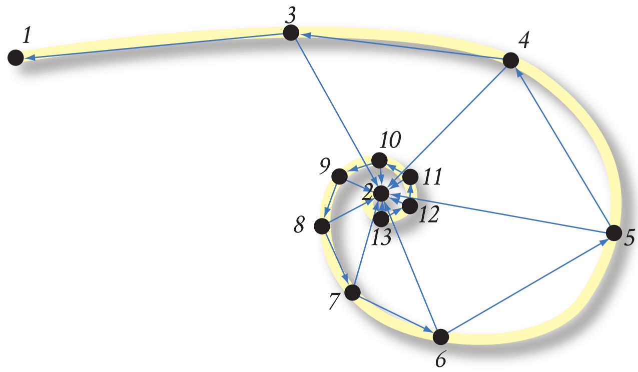


Figure 3: Point sets with bounded pacing can still have high spread, i.e. the ratio of largest to smallest interpoint distances can be $2^{O(n)}$. In the figure, each point has arrows directed at its two nearest predecessors.

The use of feature size integrals can also be used in the anisotropic setting [1]. Pav also gave anisotropic bounds on mesh size in terms of the smallest angles allowed[12].

There is a strong connection between the present work and the linear cost of *balancing* a quadtree [11]. The corners of quadtree may be viewed as a well-paced set of points (with pacing = 1). The balancing of the quadtree refines it, adding only a linear number of points.

## 3 Upper Bound

The proof of the upper bound on the feature size integral will follow a simple pattern. First, we prove a bound for inputs consisting of only two points (Lemma 2). Then, we bound the change in the integral upon adding a single new point (Lemma 3). Finally, we apply this Lemma inductively to get the final bound (Theorem 4).

**Lemma 2 (Just two points)** *If $P = \{p, q\}$ and $\Omega \subseteq \mathbf{ball}(p, c|p-q|)$ for some constant $c > 2$, then*

$$\int_\Omega \frac{dx}{\mathbf{f}_P(x)^d} \leq \mathbb{V}_d(1 + d\ln(2c)).$$

**Proof.** For all $x \in \mathbb{R}^d$, $\mathbf{f}_P(x) \geq \max\{\frac{1}{2}|p-q|, |x-p|\}$. So, we can rewrite the integral in polar coordinates (centered at $p$) and bound it as follows.

$$\begin{aligned}\int_\Omega \frac{dx}{\mathbf{f}_P(x)^d} &\leq d\mathbb{V}_d \left( \int_0^{\frac{1}{2}|p-q|} \frac{r^{d-1}dr}{(\frac{1}{2}|p-q|)^d} + \int_{\frac{1}{2}|p-q|}^{c|p-q|} \frac{r^{d-1}dr}{r^d} \right) \\ &\leq d\mathbb{V}_d \left( \frac{1}{d} + \ln\left(\frac{2c|p-q|}{|p-q|}\right)\right) \\ &= \mathbb{V}_d(1 + d\ln(2c)).\end{aligned}$$

◻

We now bound the change in the feature size integral induced by the addition of a single point.

**Lemma 3 (One more point upper bound)** *Let $P$ be a point set and let $P' = P \cup \{q\}$. If $\phi = \frac{\mathbf{f}_P(q)}{\mathbf{f}_{P'}(q)}$ then*

$$\int_\Omega \left(\frac{1}{\mathbf{f}_{P'}(x)^d} - \frac{1}{\mathbf{f}_P(x)^d}\right) dx \leq \mathbb{V}_d\left(1 + d\ln(3d\phi)\right).$$

**Proof.** Let $U$ be the subset of $\mathbb{R}^d$ where $\mathbf{f}_P \neq \mathbf{f}_{P'}$. Clearly, the integral is 0 outside $U$, so we can restrict our attention to $U$. Let $R = \mathbf{f}_{P'}(q)$; this is the distance from $q$ to the nearest point of $P$. This implies that $R\phi = \mathbf{f}_P(q)$. For all points $x$ in the ball $B = \mathbf{ball}(q, \frac{R}{2})$, $\mathbf{f}_{P'}(x) \geq \frac{R}{2}$, so

$$\int_B \left(\frac{1}{\mathbf{f}_{P'}(x)^d} - \frac{1}{\mathbf{f}_P(x)^d}\right) dx \leq \int_B \left(\frac{2}{R}\right)^d dx = \mathbb{V}_d.$$

The definitions imply the following bounds for any $x \in U$:

$$\begin{aligned}\mathbf{f}_P(x) &\leq |x-q| + R\phi, \\ \mathbf{f}_{P'}(x) &\geq |x-q|.\end{aligned}$$

The upper bound follows because $\mathbf{f}_P$ is 1-Lipschitz. The lower bound follows because $q$ must be one of the two nearest neighbors of $x$ if $\mathbf{f}_P(x) \neq \mathbf{f}_{P'}(x)$. We apply these bounds as follows.

$$\begin{aligned}&\int_{U\setminus B} \left(\frac{1}{\mathbf{f}_{P'}(x)^d} - \frac{1}{\mathbf{f}_P(x)^d}\right) dx \\ &\quad\leq \int_{U\setminus B} \left(\frac{1}{|x-q|^d} - \frac{1}{(|x-q| + R\phi)^d}\right) dx \\ &\quad\leq d\mathbb{V}_d \int_{R/2}^\infty \left(\frac{1}{r^d} - \frac{1}{(r+R\phi)^d}\right) r^{d-1}dr \\ &\quad< d\mathbb{V}_d \ln(3d\phi).\end{aligned}$$

We have extended the integral over all of $\mathbb{R}^d \setminus B$ (the function is nonnegative) and rewrote it in polar coordinates. The final inequality follows from a straightforward calculus exercise (the full proof may be found in Lemma 7 below). To bound the integral over all of $\Omega$, we simply add the bounds on the integral over $B$ and $U \setminus B$. ◻

**Theorem 4 (Upper bound)** *Let $P = \{p_1 \ldots, p_n\}$ be an ordered set of points such that $|p_1 - p_2| =$ $\mathbf{diameter}(P)$. Let $\Omega \subseteq \mathbf{ball}(p_1, c\,\mathbf{diameter}(P))$ for some constant $c > 1$ be the bounding region. Then,*

$$\int_\Omega \frac{dx}{\mathbf{f}_P(x)^d} < \mathbb{V}_d \left(1 + d\ln(2c) + \sum_{i=3}^n (1 + d\ln(3d\phi_i))\right).$$

**Proof.** We rewrite the integral as a telescoping sum:

$$\int_\Omega \frac{dx}{\mathbf{f}_P(x)^d} = \int_\Omega \frac{dx}{\mathbf{f}_{P_2}(x)^d} + \sum_{i=3}^n \left(\int_\Omega \frac{dx}{\mathbf{f}_{P_i}(x)^d} - \int_\Omega \frac{dx}{\mathbf{f}_{P_{i-1}}(x)^d}\right).$$

The bounds from Lemmas 2 and 3 complete the proof. ◻

## 4 Lower Bound

The proof of the lower bound will be similar to the proof of the upper bound in that we will use the pacing of a single new point to bound the change in the feature size integral.

**Lemma 5 (One more point lower bound)** *Let $P$ be a set of at least 2 points and let $P' = P \cup \{q\}$ for some $q \in \mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be a set containing $\mathbf{ball}(q, \mathbf{diameter}(P'))$. If $\phi = \frac{\mathbf{f}_P(q)}{\mathbf{f}_{P'}(q)}$ then*

$$\int_\Omega \left(\frac{1}{\mathbf{f}_{P'}(x)^d} - \frac{1}{\mathbf{f}_P(x)^d}\right) dx \geq \frac{\mathbb{V}_d}{2^d}\left(d\ln\frac{\phi}{3} - 1\right)$$

**Proof.** The bound is trivial if $\phi \leq 3$, so we may assume that $\phi > 3$. Let $R = \mathbf{f}_{P'}(q)$. Since $\mathbf{f}_P \geq \mathbf{f}_{P'}$, it will suffice to prove a lower bound on the change in the feature size integral over the subset $U = \{x : R \leq |x-q| \leq \frac{R\phi}{3}\} \subseteq \Omega$. For all $x \in U$,

$$\begin{aligned}\mathbf{f}_P(x) &\geq \frac{2R\phi}{3}, \text{ and} \\ \mathbf{f}_{P'}(x) &\leq R + |x-q| \leq 2|x-q|.\end{aligned}$$

The lower bound follows because there is at most one point of $P$ in the interior of $\mathbf{ball}(q, R\phi)$. The upper bound follows because $\mathbf{f}_{P'}(x)$ is 1-Lipschitz. We apply these bounds and convert to polar coordinates:

$$\begin{aligned}
&\int_U \left( \frac{1}{\mathbf{f}_{P'}(x)^d} - \frac{1}{\mathbf{f}_P(x)^d} \right) dx \\
&\geq \int_U \left( \frac{1}{(2|x-q|)^d} - \left( \frac{3}{2R\phi} \right)^d \right) dx \\
&> \left( d\mathbb{V}_d \int_R^{\frac{R\phi}{3}} \frac{r^{d-1}}{(2r)^d} dr \right) - \frac{\mathbb{V}_d}{2^d} \\
&= \frac{\mathbb{V}_d}{2^d} \left( d \ln \frac{\phi}{3} - 1 \right)
\end{aligned}$$

□

When we apply the preceding lemma to a set of $n$ points, we get the following lower bound.

**Theorem 6 (Lower bound)** *Let $P = \{p_1 \dots, p_n\}$ be an ordered set of points. If $\Omega \subset \mathbb{R}^d$ is a set containing* $\mathbf{ball}(p, 2\mathbf{diameter}(P))$ *for some $p \in P$, then*

$$\int_\Omega \frac{dx}{\mathbf{f}_P(x)^d} > \frac{\mathbb{V}_d}{2^d} \sum_{i=3}^{n} \left( d \ln \frac{\phi_i}{3} - 1 \right).$$

## 5 Some directions for future work

The work of Ruppert on optimal meshing has been extended to feature size functions that also take into account input features beyond just point sets, including piecewise linear or even piecewise smooth complexes [4]. One direction for future work is to extend these methods for bounding the feature size to these settings as well.

It would also be interesting to extend this approach to anisotropic case, such as in [1]. In that setting, it is not known how to relax the quality constraints to guarantee a linear size mesh.

Moreover, since Theorem 1 describes the cost of adding a single point, it makes sense to apply these analytic techniques to dynamic meshing problems.

## References

[1] Jean-Daniel Boissonnat, Camille Wormser, and Mariette Yvinec. Anisotropic Delaunay Mesh Generation. Rapport de recherche RR-7712, INRIA, August 2011.

[2] Frédéric Chazal, David Cohen-Steiner, and Quentin Mérigot. Geometric inference for probability measures. *Foundations of Computational Mathematics*, 11:733–751, 2011.

[3] Frédéric Chazal and André Lieutier. Topology guaranteeing manifold reconstruction using distance function to noisy data. In *Proceedings of the 22nd ACM Symposium on Computational Geometry*, 2006.

[4] Tamal K. Dey and Joshua A. Levine. Delaunay meshing of piecewise smooth complexes without expensive predicates. *Algorithms*, 2(4):1327–1349, 2009.

[5] Jeff Erickson. Nice point sets can have nasty Delaunay triangulations. In *Proceedings of the 17th ACM Symposium on Computational Geometry*, pages 96–105, 2001.

[6] Jeff Erickson. Dense point sets have sparse delaunay triangulations or "... but not too nasty". *Discrete & Computational Geometry*, 33:83–115, 2005.

[7] Benoît Hudson, Gary L. Miller, Steve Y. Oudot, and Donald R. Sheehy. Topological inference via meshing. In *Proceedings of the 26th ACM Symposium on Computational Geometry*, pages 277–286, 2010.

[8] Benoît Hudson, Gary L. Miller, Todd Phillips, and Donald R. Sheehy. Size complexity of volume meshes vs. surface meshes. In *SODA: ACM-SIAM Symposium on Discrete Algorithms*, 2009.

[9] Gary L. Miller, Todd Phillips, and Donald R. Sheehy. Linear-size meshes. In *CCCG: Canadian Conference in Computational Geometry*, pages 175–178, 2008.

[10] Gary L. Miller, Todd Phillips, and Donald R. Sheehy. Beating the spread: Time-optimal point meshing. In *SOCG: Proceedings of the 26th ACM Symposium on Computational Geometry*, pages 321–330, 2011.

[11] Doug Moore. The cost of balancing generalized quadtrees. In *SMA '95: Proceedings of the Third Symposium on Solid Modeling and Applications*, pages 305–312, 1995.

[12] Steven E. Pav. An anisotropic cardinality bound for triangulations. In *CCCG: The Canadian Conference on Computational Geometry*, pages 95–98, 2004.

[13] Jim Ruppert. A Delaunay refinement algorithm for quality 2-dimensional mesh generation. *J. Algorithms*, 18(3):548–585, 1995.

[14] Donald R. Sheehy. *Mesh Generation and Geometric Persistent Homology.* PhD thesis, Carnegie Mellon University, 2011.

[15] Dafna Talmor. *Well-Spaced Points for Numerical Methods.* PhD thesis, Carnegie Mellon University, 1997.

## A A bit of calculus

**Lemma 7** *Given positive constants $\phi \geq 1$ and $R$,*

$$\int_{R/2}^{\infty} \left( \frac{1}{r^d} - \frac{1}{(r+R\phi)^d} \right) r^{d-1} dr < \ln(3d\phi).$$

**Proof.** We bound this integral using the change of variables $u = \frac{R\phi}{r} + 1$ as follows.

$$\begin{aligned}
\int_{R/2}^{\infty} \left( \frac{1}{r^d} - \frac{1}{(r+R\phi)^d} \right) r^{d-1} dr &= \int_1^{1+2\phi} \left( \frac{u^d - 1}{u^d(u-1)} \right) du \\
&= \sum_{i=0}^{d-1} \int_1^{1+2\phi} u^{i-d} du \\
&< \ln(1+2\phi) + \sum_{i=0}^{d-2} \frac{1}{d-i-1} \\
&< \ln(1+2\phi) + \ln d \\
&\leq \ln(3d\phi).
\end{aligned}$$

□